Yu. A AMENZADE

THEORY OF ELASTICITY

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Ю. А. АМЕНЗАДЕ

Теория упругости

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Yu. A. AMENZADE

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M. KONYAEVA

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Notation

A	external work, Hölder's constant, work of deformation
A_n, A^n	covariant and contravariant vectors
A_{mn}, A^{mn}, A^n_m	covariant, contravariant, and mixed tensors
$A_{mn,\beta}$, $A^{mn}_{,\beta}$	covariant derivatives of tensors A_{mn}, A^{mn}, respectively
a	thermal diffusivity
a_m	components of vector a in rectangular Cartesian co-ordinate system
a_{e_k}, a_{e^k}	projections of vector a on e_k, e^k, respectively
a_{x^k}	physical projections of vector a
$B_{\beta,n}$, $B^{\beta}_{,n}$	covariant derivatives of vectors B_β, B^β, respectively
C^{ijkl}	elastic coefficients
c	height, phase velocity, specific heat
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
c_1, c_2	wave velocities
D	flexural rigidity of plate, torsional rigidity
E	modulus of elasticity in tension or compression
e	unit vector
e_n	extension along co-ordinate line
e_n, e^n	covariant and contravariant base vectors
e_{nk}	components of small strain tensor
F	Helmholtz free energy
F	body force vector per unit mass
F^*	body force vector per unit volume
$F(z)$	complex torsion function
f	invariant or scalar

G_{kn}	cofactor of element g_{kn}
g	acceleration of gravity, determinant
g_{nk}, g^{nk}, g_k^n	covariant, contravariant, and mixed metric tensors
H_k	scale factors
h	height, thickness
I	moment of inertia, Reissner's functional
I_0	polar moment of inertia
$I_0\,(ikr)$	Bessel function of zero order
$I_1\,(ikr)$	Bessel function of first order
I_1, I_2, I_3	invariants of stress and strain tensors
i	imaginary unity
$\mathbf{i}_m$	unit vectors of rectangular Cartesian coordinate system
K	bulk modulus, curvature of elastic line, kinetic energy
k	surface heat transfer coefficient
$\mathbf{L}$	resultant moment
l	arc length, length
l_k	direction cosines
M	moment of couple
$\mathbf{W}_n$	couple-stress vector
M_t	twisting moment (torque)
m	mass
$\mathbf{n}$	unit normal vector
$\mathbf{P}$	surface force vector
p	load intensity, pressure
p_{mn}	affine orthogonal tensor
$p_{x^ix^k}$	physical projections of tensor p_{mn}
Q	heat quantity, shearing force
q	frequency, load intensity
R	external work, radius, radius of curvature, strain energy
R^*	complementary energy
R_{prst}	Riemann-Christoffel tensor
Re	real part
r	radius
$\mathbf{r}$	radius vector

r, θ	plane polar co-ordinates
r, φ, x_3	cylindrical co-ordinates
r, φ, ψ	spherical co-ordinates
S	entropy, static moment of area
S^+, S^-	upper and lower half-plane, respectively
T	absolute temperature, tension
T_n	stress vector
T_n^m	contravariant components of stress vector
t	point of curve, time
U	internal energy
u	displacement vector
V	resultant vector
v	velocity, volume
W	acceleration vector
w	deflection of plate
x_k	rectangular Cartesian co-ordinates
x^k	curvilinear co-ordinates
α	coefficient of linear thermal expansion, Hölder's exponent
Γ_{nk}	Christoffel symbols
γ	specific weight (weight per unit volume)
Δ	Laplacian operator
δ	thickness
δ_{rk}	Kronecker symbols
$\delta\,(x_k)$	Dirac function
$\frac{\delta B_\beta}{\delta s}$, $\frac{\delta B^\beta}{\delta s}$, $\frac{\delta A_{mn}}{\delta s}$	absolute derivatives of vectors B_β, B^β, and tensor A_{mn}, respectively
ε_{ijk}	ε-tensor
ε_{nk}	components of finite strain tensor
ε_1, ε_2, ε_3	principal extensions
θ	volume strain
ϑ	angle
λ^*	thermal conductivity
λ, μ	Lamé's elastic constants
ν	Poisson's ratio
ξ, η, ζ	rectangular co-ordinates
Π	potential energy

ρ	density
$\sigma = \frac{1}{3}\sigma_{kk}$	mean (hydrostatic) pressure
σ_{mk}	components of affine orthogonal stress tensor
σ^{km}	contravariant components of stress tensor
$\boldsymbol{\sigma}_n$	normal stress vector
$\sigma_1, \sigma_2, \sigma_3$	principal normal stresses
τ	angle of twist per unit length, volume
$\boldsymbol{\tau}_n$	shearing stress vector
Φ	Airy's stress function
$\Phi(x_1, x_2)$	stress function in torsion or Prandtl's stress function
$\Phi(z), \Psi(z)$	complex potentials
φ	polar angle
$\varphi(x_1, x_2)$	Saint Venant's torsion function
$\varphi(z), \psi(z), \chi(z)$	analytic functions of complex variable z
Ψ	flexure function
ω	angular velocity, surface area
ω_{kn}	rotation tensor
∇	Hamiltonian operator

Introduction

The theory of elasticity is concerned with the mechanics of deformable media which, after the removal of the forces producing deformation, completely recover their original shape and give up all the work expended in the deformation.

The first attempts to develop the theory of elasticity on the basis of the concept of a continuous medium, which enables one to ignore its molecular structure and describe macroscopic phenomena by the methods of mathematical analysis, date back to the first half of the eighteenth century.

The fundamental contribution to the classical theory was made by R. Hooke, C. L. M. H. Navier, A. L. Cauchy, G. Lamé, G. Green, B. P. E. Clapeyron. In 1678 Hooke established a law linearly connecting stresses and strains.

After Navier established the basic equations in 1821 and Cauchy developed the theory of stress and strain, of great importance in the development of elasticity theory were the investigations of B. de Saint Venant. In his classical work on the theory of torsion and bending Saint Venant gave the solution of the problems of torsion and bending of prismatic bars on the basis of the general equations of the theory of elasticity. In these investigations Saint Venant devised a semi-inverse method for the solution of elasticity problems, formulated the famous Saint Venant's principle, which enables one to obtain the solution of elasticity problems. Since then much effort has been made to develop the theory of elasticity and its applications, a number of general theorems have been proved, the general methods for the integration of differential equations of equilibrium and motion have been proposed, many special problems of fundamental interest have been solved. The development of new fields of engineering demands deeper and more extensive studies of the theory of elasticity. High velocities call for the formulation and solution of complex vibrational problems. Lightweight metallic structures draw particular attention to the question of elastic stability. The concentration of stress entails dangerous consequences, which cannot safely be ignored.

CHAPTER 1

Elements of tensor calculus

Many problems of mechanics, theoretical physics, and other sciences lead to the concept of a tensor. This concept is of a more complicated nature than the concept of a vector. The definition of a vector as a directed segment does not allow one to pass to the concept of a tensor by a natural generalization. We shall therefore try to give a definition of a vector, equivalent to the former one, such that its generalization will lead to the concept of a tensor, which cannot be explained by means of a simple geometrical image. To do this, we have to introduce into consideration arbitrary curvilinear co-ordinates. With reference to these co-ordinates we shall give a definition of a vector, and subsequently a definition of a tensor as some object that is not altered by a change of the co-ordinate system.

The advantage of tensor calculus in continuum mechanics is particularly apparent when we deal with arbitrary co-ordinate systems. In the following discussion we shall restrict our attention to a three-dimensional Euclidean space in which the position of each point is determined by three numbers, co-ordinates. Here we shall present some basic data from tensor calculus. The presentation makes no claim to be complete or rigorous; a summary of definitions and formulas is given which will be referred to in what follows.

Denote the curvilinear co-ordinates of some point by x^1, x^2, x^3 and introduce new co-ordinates of this point $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$ connected with the old ones by the relations

$$x^k = x^k(\bar{x}^1, \bar{x}^2, \bar{x}^3) \quad (k = 1, 2, 3), \tag{1.1}$$

which are called the formulas of transformation of co-ordinates.

Suppose that all functions x^k in the given range of co-ordinates $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$ are single valued, continuous and have continuous partial derivatives of the first order, and the Jacobian is different from zero. From (1.1) we then find a transformation of co-ordinates which is inverse to transformation (1.1)

$$\bar{x}^k = \bar{x}^k(x^1, x^2, x^3) \quad (k = 1, 2, 3), \tag{1.2}$$

If any two of the three co-ordinates are fixed and the third one is varied continuously, we obtain a line which is called a co-ordinate line. We assume that at each point of space there pass three co-ordinate lines not lying in the same plane. It may be proved that this requirement is always fulfilled if the Jacobian of transformation (1.1) is not zero.

In the particular case, when we transform from one rectilinear rectangular co-ordinate system $ox_1x_2x_3$ to another system $ox'_1x'_2x'_3$, instead of (1.1) we have

$$x_k = \sum_{n=1}^{3} \alpha_{kn} x'_n \quad (k = 1,\ 2,\ 3), \tag{1.3}$$

where α_{kn} are the cosines of the angles between the axes of the co-ordinate systems $ox_1x_2x_3$ and $ox'_1x'_2x'_3$.

Here and henceforth, we agree, for shortness in writing, to omit the summation sign in (1.3) assuming that the repeated index must be summed from $n = 1$ to $n = 3$. We shall no longer mention that we have three formulas ($k = 1, 2, 3$). Relations (1.3) are then written as

$$x_k = \alpha_{kn} x'_n. \tag{1.4}$$

A transformation of the form (1.4) is said to be affine orthogonal.

1. SCALARS, VECTORS, AND TENSORS

Suppose we have a quantity $f(x^1, x^2, x^3)$ in some coordinate system x^n ($n = 1, 2, 3$), and a quantity $\bar{f}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ in the system $\bar{x}^k$; if under transformation (1.1) the values of these quantities at the same points are equal, the quantity f is called an invariant, or a scalar. Examples of scalar quantities are density, temperature.

Suppose we have a set of three quantities A^n in some co-ordinate system x^n ($n = 1, 2, 3$), and a set $\bar{A}^k$ in the system $\bar{x}^k$ ($k = 1, 2, 3$); if under the transformation of co-ordinates (1.2) the quantities $\bar{A}^k$ are determined by the formulas

$$\bar{A}^k = A^n \frac{\partial \bar{x}^k}{\partial x^n}, \tag{1.5}$$

the set of three quantities A^n is called a contravariant vector, and the quantities A^n are called its components. As in formula (1.4), the summation in formula (1.5) is carried out with respect to the index n, which appears twice. It is easy to see that the set of three differentials of the co-ordinates forms a contravariant vector.

Indeed, from formulas (1.2) we have

$$d\bar{x}^k = \frac{\partial \bar{x}^k}{\partial x^1} dx^1 + \frac{\partial \bar{x}^k}{\partial x^2} dx^2 + \frac{\partial \bar{x}^k}{\partial x^3} dx^3 = dx^n \frac{\partial \bar{x}^k}{\partial x^n}. \tag{1.6}$$

Comparison of formulas (1.6) and (1.5) shows that $d\bar{x}^n$ are the components of a contravariant vector.

Suppose we have a set of three quantities A_n in some co-ordinate system x^n, and a set $\bar{A}_k$ in the system $\bar{x}^k$; if under the transformation of co-ordinates (1.1) the quantities $\bar{A}_k$ are determined by the formulas

$$\bar{A}_k = A_n \frac{\partial x^n}{\partial \bar{x}^k}, \tag{1.7}$$

the set of three quantities A_n is called a covariant vector, and the quantities A_n are called its components. It can easily be verified that in the case of the affine orthogonal transformation the definitions of a contravariant and a covariant vector are identical. Indeed, by solving Eqs. (1.4) for

$$x'_n = \alpha_{kn} x_k, \tag{1.8}$$

from (1.4) and (1.8) we find

$$\frac{\partial x_k}{\partial x'_n} = \frac{\partial x'_n}{\partial x_k} = \alpha_{kn}. \tag{1.9}$$

The last relations show that the transformation formulas (1.5) and (1.7) coincide, i.e., we have

$$a'_k = a_n \alpha_{kn}.$$

The set of quantities a_n ($n = 1, 2, 3$) is called an affine orthogonal vector.

Suppose we have a set of nine quantities A^{mn} in some co-ordinate system x^α, and a set $\bar{A}^{ik}$ in the system $\bar{x}^i$; if under the transformation of co-ordinates (1.2) the quantities $\bar{A}^{ik}$ are determined by the formulas

$$\bar{A}^{ik} = A^{mn} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial \bar{x}^k}{\partial x^n}, \tag{1.10}$$

the set of nine quantities A^{mn} is called a contravariant tensor of rank two, and the quantities A^{mn} are called its components.

In formulas (1.10) a double summation must be performed for all values of the repeated indices n and m ($m, n = 1, 2, 3$). If

$$\bar{A}_{ik} = A_{mn} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^k}, \tag{1.11}$$

the set of nine quantities A_{mn} is called a covariant tensor of rank two.

If

$$\bar{A}_i^k = A_m^n \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^n}, \tag{1.12}$$

the set of nine quantities A_m^n is called a mixed tensor of rank two. In the case of the affine orthogonal transformation the definitions of a contravariant, a covariant, and a mixed tensor are identical by virtue of (1.9), i.e.,

$$p'_{ik} = p_{mn}\alpha_{mi}\alpha_{nk}. \tag{1.13}$$

The set of nine quantities p_{nm} is called an affine orthogonal tensor of rank two.

Suppose that in any co-ordinate system we have a set of the following nine numbers:

$$\delta_m^n = \begin{cases} 1 \text{ when } m = n, \\ 0 \text{ when } m \neq n. \end{cases} \tag{1.14}$$

We shall show that δ_m^n, called the Kronecker symbols, are the components of a mixed tensor of rank two; to do this, we must show that the δ_m^n satisfy formulas (1.12)

$$\bar{\delta}_i^k = \delta_m^n \frac{\partial \bar{x}^k}{\partial x^n} \frac{\partial x^m}{\partial \bar{x}^i}. \tag{1.15}$$

By (1.14), we have

$$\delta_m^n \frac{\partial \bar{x}^k}{\partial x^n} \frac{\partial x^m}{\partial \bar{x}^i} = \frac{\partial \bar{x}^k}{\partial x^n} \frac{\partial x^n}{\partial \bar{x}^i}. \tag{1.16}$$

Substituting (1.1) in (1.2), we obtain

$$\bar{x}^k = \bar{x}^k (x^1 (\bar{x}^1, \bar{x}^2, \bar{x}^3), x^2 (\bar{x}^1, \bar{x}^2, \bar{x}^3), x^3 (\bar{x}^1, \bar{x}^2, \bar{x}^3)). \tag{1.17}$$

By differentiating both sides of (1.17), we find

$$\frac{\partial \bar{x}^k}{\partial \bar{x}^i} = \frac{\partial \bar{x}^k}{\partial x^n} \frac{\partial x^n}{\partial \bar{x}^i}.$$

On the other hand,

$$\frac{\partial \bar{x}^k}{\partial \bar{x}^i} = \bar{\delta}_i^k. \tag{1.18}$$

On comparing (1.18) with (1.16) we obtain (1.15), which was to be proved.

Tensors of higher rank are defined in an analogous way. Thus, if

$$\bar{A}_{ikr} = A_{mns} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^r},$$

the set of 27 quantities A_{mns} is called a covariant tensor of rank three (the number of components of a tensor is determined by the number of dimensions of the space to a power equal to the rank of the tensor).

Consider a contravariant and a covariant tensor of rank two, A^{mn}, A_{mn}. If, when the indices of A^{mn}, A_{mn} are interchanged, the following relations are valid:

$$A^{nm} = A^{mn}, \quad A_{nm} = A_{mn}, \tag{1.19}$$

the tensors are said to be, respectively, symmetric contravariant and symmetric covariant. If, when the indices are interchanged, the following relations are valid:

$$A^{nm} = -A^{mn}, \quad A_{nm} = -A_{mn}, \tag{1.20}$$

the tensors are said to be, respectively, antisymmetric contravariant and antisymmetric covariant.

2. ADDITION, MULTIPLICATION, AND CONTRACTION OF TENSORS. THE QUOTIENT LAW OF TENSORS

(a) *Addition.* The operation of addition applies only to tensors having the same number of lower and upper indices (i.e., to tensors of the same rank and type). If we are given two tensors of the same rank and type, and if we sum algebraically each component of the first tensor and the corresponding component of the second tensor, we obviously obtain a tensor of the same rank and type as the original tensors. This operation is called addition, and the resulting tensor is called the sum of the two tensors.

(b) *Multiplication.* Let us define the product of two tensors of any rank and type. By multiplying each component of the first tensor by each component of the second tensor, we obtain a tensor whose rank equals the sum of the ranks of the two original tensors. This operation is called multiplication, and the resulting tensor is called the product of the two tensors. For definiteness, we assume that the multiplication in question is that of a contravariant tensor of rank two A^{mn} by a tensor of rank three B^r_{st} (B^r_{st} is once contravariant and twice covariant). We then obtain a tensor C^{mnr}_{st} whose components are determined by the formulas

$$C^{mnr}_{st} = A^{mn} B^r_{st}. \tag{1.21}$$

This is a tensor of rank five (three times contravariant, twice covariant).

The operations of addition and multiplication can be extended to any number of tensors.

(c) *Contraction* (reduction of indices). The operation of contraction applies only to mixed tensors; we shall illustrate this by a series of examples. Let us take, for example, a tensor of rank four A^n_{stm}, which has one contravariant index and three covariant indices. Putting now $m = n$, we obtain the tensor $B_{st} = A^n_{stn}$, in which n is a repeated index; in accordance with our convention it must be summed from 1 to 3. As a result we obtain a covariant tensor of rank two, i.e., a tensor whose rank is two less than that of the original tensor. The operation of contraction, obviously, cannot be repeated any more in this example.

Let us now take a tensor of rank five A^{mn}_{str} and contract with respect to any pair of indices, one of which is a superscript and the other a subscript. If, for example, we put $s = m$, we obtain a tensor of rank three $B^{n}_{tr} = A^{mn}_{mtr}$. We can contract once more with respect to r and n, obtaining the covariant vector $C_t = A^{mn}_{mtn}$. From a tensor of rank five A^{mnr}_{st}, after double reduction of indices, we obtain the contravariant vector $D^n = A^{mnr}_{rm}$. If in a tensor of rank four A^{mn}_{st} the contraction is carried out twice, we obtain the scalar (invariant) $f_1 = A^{mn}_{mn}$ or $f_2 = A^{nm}_{nm}$. In the case of affine orthogonal tensors the operation of contraction can be carried out with respect to any two indices since there is no difference whatsoever between contravariant and covariant affine orthogonal tensors.

By contracting an affine orthogonal tensor p_{mn} with respect to the indices m and n, we obtain the invariant

$$C = p_{mm} = p_{11} + p_{22} + p_{33}. \tag{1.22}$$

A combination of the operations of multiplication and contraction is called scalar (inner) multiplication. The operation of scalar multiplication of two tensors reduces first to their multiplication, and then to the contraction of the resulting tensor with respect to a superscript of one tensor and a subscript of the other. Suppose we have two tensors, A^{mn} and B^{s}_{rt}; by contracting their tensor product in four ways, we obtain a scalar product, viz. $A^{mn} B^{s}_{mt}$, $A^{mn} B^{s}_{rn}$, $A^{rn} B^{s}_{rt}$, $A^{mt} B^{s}_{rt}$. A scalar product of a contravariant vector A^m and a covariant vector B_n is the invariant $A^n B_n$, which can obviously be termed the scalar product of the vectors A^m and B_n. In the case of affine orthogonal vectors a_n and b_m we obtain the scalar product of these vectors, $\mathbf{a} \cdot \mathbf{b} = a_n b_n$.

(d) *The quotient law of tensors.* Suppose some tensor is given, say A^{k}_{mn}. We set into correspondence with the covariant indices of this tensor arbitrary contravariant vectors u^{α} and v^{β}, and with the contravariant index a covariant vector w_{γ}. If the product $A^{k}_{mn} u^{\alpha} v^{\beta} w_{\gamma}$ representing a tensor of rank six is contracted with respect to the indices m and α, n and β, k and γ, we obtain the invariant

$$f = A^{k}_{mn} u^{m} v^{n} w_{k}. \tag{1.23}$$

Further, suppose we are given two tensors, A^{k}_{mn} and $B^{\alpha\beta}$. If the product $A^{k}_{mn} B^{\alpha\beta}$, which represents a tensor of rank five, is contracted with respect to the indices m and α, n and β, we have the contravariant vector

$$A^{k}_{\alpha\beta} B^{\alpha\beta} = C^{k}. \tag{1.24}$$

Thus, the operation of multiplication of tensors gives again a tensor. We may now inquire whether a certain system of quantities is a tensor if its product with a tensor gives a tensor. There is a theorem on that score which provides a means for easily establishing the tensor character of a given system of quantities. This theorem may be formulated as follows:

(1) if for an arbitrary choice of the vectors u^m, v^n, w_k the product (1.23) represents an invariant, then A^k_{mn} is a tensor;

(2) if for an arbitrary choice of the tensor $B^{\alpha\beta}$ the product (1.24) represents a contravariant vector, then A^k_{mn} is a tensor;

(3) if the quantities A_{mn} possess the symmetry property and the product $A_{mn}u^m u^n$ is an invariant for an arbitrary vector u^k, then A_{mn} is a tensor.

To prove the theorem of the form (1), it is necessary to verify that the components A^k_{mn} satisfy the definition of a tensor.

According to the condition of the theorem, for two co-ordinate systems, $\bar x^n$ and x^n, we have $\bar f = f$, or on the basis of (1.23)

$$\bar A^{\gamma}_{\alpha\beta}\bar u^{\alpha}\bar v^{\beta}\bar w_{\gamma} = A^{\gamma}_{\alpha\beta}u^{\alpha}v^{\beta}w_{\gamma}.$$

By interchanging the co-ordinates $\bar x^n$ and x^n in formulas (1.5) and (1.7), in the system x^n we obtain

$$u^{\alpha} = \bar u^{m}\frac{\partial x^{\alpha}}{\partial \bar x^{m}}, \qquad v^{\beta} = \bar v^{n}\frac{\partial x^{\beta}}{\partial \bar x^{n}}, \qquad w_{\gamma} = \bar w_{k}\frac{\partial \bar x^{k}}{\partial x^{\gamma}}.$$

Substituting these relations in the last formula, we have

$$\bar A^{\gamma}_{\alpha\beta}\,\bar u^{\alpha}\bar v^{\beta}\bar w_{\gamma} = A^{\gamma}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial \bar x^{m}}\frac{\partial x^{\beta}}{\partial \bar x^{n}}\frac{\partial \bar x^{k}}{\partial x^{\gamma}}\,\bar u^{m}\bar v^{n}\bar w_{k}.$$

Hence,

$$\left(\bar A^{\gamma}_{\alpha\beta} - A^{k}_{mn}\frac{\partial x^{m}}{\partial \bar x^{\alpha}}\frac{\partial x^{n}}{\partial \bar x^{\beta}}\frac{\partial \bar x^{\gamma}}{\partial x^{k}}\right)\bar u^{\alpha}\bar v^{\beta}\bar w_{\gamma} = 0.$$

Since, by condition, the contravariant vectors u^{α}, v^{β} and the covariant vector w_{γ} are arbitrary, we have

$$\bar A^{\gamma}_{\alpha\beta} = A^{k}_{mn}\frac{\partial x^{m}}{\partial \bar x^{\alpha}}\frac{\partial x^{n}}{\partial \bar x^{\beta}}\frac{\partial \bar x^{\gamma}}{\partial x^{k}}.$$

Consequently, A^k_{mn} is a tensor.

To prove the theorem of the form (2), relation (1.24) is multiplied scalarly by an arbitrary covariant vector D_j; then

$$A^{k}_{\alpha\beta}B^{\alpha\beta}D_{k} = C^{k}D_{k} = f,$$

where f is an invariant. Consequently,

$$\bar A^{k}_{\alpha\beta}\bar B^{\alpha\beta}\bar D_{k} = A^{k}_{\alpha\beta}B^{\alpha\beta}D_{k}.$$

By interchanging the co-ordinates $\bar{x}^n$ and x^n in formulas (1.10) and (1.7), in the system x^n we obtain

$$B^{\alpha\beta}=\bar{B}^{mn}\frac{\partial x^\alpha}{\partial \bar{x}^m}\frac{\partial x^\beta}{\partial \bar{x}^n},\quad D_k=\bar{D}_\gamma\frac{\partial \bar{x}^\gamma}{\partial x_k}.$$

Thus,

$$\left(\bar{A}^\gamma_{\alpha\beta}-A^k_{mn}\frac{\partial x^m}{\partial \bar{x}^\alpha}\frac{\partial x^n}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\gamma}{\partial x^k}\right)\bar{B}^{mn}\bar{D}_\gamma=0.$$

From this, since the tensors B^{mn} and D_γ are arbitrary, we have

$$\bar{A}^\gamma_{\alpha\beta}=A^k_{mn}\frac{\partial x^m}{\partial \bar{x}^\alpha}\frac{\partial x^n}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\gamma}{\partial x^k},$$

which was to be proved.

To prove the theorem of the form (3), the contravariant vector is represented as $u^k = v^k + w^k$. Then

$$A_{mn}u^m u^n=A_{mn}v^m v^n+A_{mn}w^m w^n+A_{mn}v^m w^n+A_{mn}v^n w^m.$$

Since, by the condition of the theorem, $A_{mn}u^m u^n$, $A_{mn}v^m v^n$, and $A_{mn}w^m w^n$ are invariants and since, by virtue of the symmetry of the quantities A_{mn}, we have $A_{mn}v^m w^n = A_{mn}v^n w^m$, it follows that $A_{mn}v^m w^n$ is an invariant. Then, noting that v^m and w^n are arbitrary vectors, we conclude from the theorem of the form (1) that A_{mn} is a covariant tensor of rank two.

3. THE METRIC TENSOR

Consider two infinitely close points A (x^1, x^2, x^3) and A_1 $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ in space. These points define an infinitesimal vector $\boldsymbol{dr}$, which is independent of the choice of co-ordinate system. Let the length of the vector $\boldsymbol{dr}$ be denoted by ds. If $\boldsymbol{e}$ defines, by condition, the unit vector directed along the straight line AA_1, then

$$\boldsymbol{dr} = ds\boldsymbol{e}. \tag{1.25}$$

From the point A (x^1, x^2, x^3) we draw co-ordinate lines which do not lie in the same plane and are not, in general, orthogonal. Denote by $\boldsymbol{e}_n$ a system of vectors, not of unit length, directed along the tangents to the co-ordinate lines; then

$$\boldsymbol{dr}_1 = dx^1\boldsymbol{e}_1,\quad \boldsymbol{dr}_2 = dx^2\boldsymbol{e}_2,\quad \boldsymbol{dr}_3 = dx^3\boldsymbol{e}_3,$$

where $\boldsymbol{dr}_1$, $\boldsymbol{dr}_2$, $\boldsymbol{dr}_3$ are infinitesimal vectors defining a parallelepiped whose diagonal is the vector $\boldsymbol{dr}$, i.e.,

$$\boldsymbol{dr} = ds\boldsymbol{e} = dx^n\boldsymbol{e}_n. \tag{1.26}$$

From this, according to the rules for the scalar multiplication of vectors, we find

$$ds^2=(dx^n\boldsymbol{e}_n\cdot dx^k\boldsymbol{e}_k)=g_{nk}\,dx^n\,dx^k, \tag{1.27}$$

where

$$g_{nk}=(\boldsymbol{e}_n\cdot\boldsymbol{e}_k)\quad(n,\ k=1,\ 2,\ 3). \tag{1.28}$$

The system of vectors $\boldsymbol{e}_n$ is called the covariant base of a co-ordinate system.

The coefficients g_{nk} in the quadratic form of the differentials dx^m, as seen from (1.28), form a symmetric matrix ($g_{nk} = g_{kn}$). Thus, by the quotient theorem, g_{nk} are the components of a covariant tensor, called the covariant metric tensor.

When using curvilinear co-ordinates, it is advisable to introduce, along with the fundamental base $\boldsymbol{e}_n$, the reciprocal contravariant base $\boldsymbol{e}^k$, i.e., a triplet of vectors $\boldsymbol{e}^k$ connected with the fundamental vectors $\boldsymbol{e}_n$ by the formulas

$$\boldsymbol{e}_n\cdot\boldsymbol{e}^k=\delta_n^k, \tag{1.29}$$

where δ_n^k are the Kronecker symbols.

To do this, it is sufficient to put

$$\begin{gathered}\boldsymbol{e}^1=\frac{\boldsymbol{e}_2\times\boldsymbol{e}_3}{g},\qquad \boldsymbol{e}^2=\frac{\boldsymbol{e}_3\times\boldsymbol{e}_1}{g},\qquad \boldsymbol{e}^3=\frac{\boldsymbol{e}_1\times\boldsymbol{e}_2}{g},\\ g=\boldsymbol{e}_1\cdot(\boldsymbol{e}_2\times\boldsymbol{e}_3).\end{gathered} \tag{1.30}$$

From (1.29) it also follows that

$$\begin{gathered}\boldsymbol{e}_1=\frac{\boldsymbol{e}^2\times\boldsymbol{e}^3}{g_1},\qquad \boldsymbol{e}_2=\frac{\boldsymbol{e}^3\times\boldsymbol{e}^1}{g_1},\qquad \boldsymbol{e}_3=\frac{\boldsymbol{e}^1\times\boldsymbol{e}^2}{g_1},\\ g_1=\boldsymbol{e}^1\cdot(\boldsymbol{e}^2\times\boldsymbol{e}^3).\end{gathered} \tag{1.31}$$

Thus, $\boldsymbol{e}_n$ is perpendicular to the ($\boldsymbol{e}^k$, $\boldsymbol{e}^m$) plane.

If the co-ordinate system is orthogonal, it is obvious [see (1.30) and (1.31)] that the base vectors $\boldsymbol{e}_n$ and $\boldsymbol{e}^k$ coincide in direction, but their magnitudes are in general different.

We represent the vector $\boldsymbol{e}_n$ as a linear combination of the vectors $\boldsymbol{e}^k$:

$$\boldsymbol{e}_n=C_{nk}\boldsymbol{e}^k\quad(k,\ n=1,\ 2,\ 3).$$

Taking into account relations (1.28) and (1.29), we obtain $C_{nk} = g_{nk}$; consequently, $\boldsymbol{e}_n = g_{nk}\boldsymbol{e}^k$.

From this, by Crammer's rule, we find

$$\boldsymbol{e}^k=\frac{G_{nk}}{g}\boldsymbol{e}_n=g^{nk}\boldsymbol{e}_n\quad(g\neq 0). \tag{1.32}$$

Here G_{nk} is the cofactor of the element g_{kn} in the determinant g.

From (1.32), on the basis of (1.29), we have

$$\boldsymbol{e}^k\cdot\boldsymbol{e}^n=\frac{G_{nk}}{g}=g^{nk}. \tag{1.33}$$

The g^{nk} are symmetric. Substituting (1.32) in (1.29), we have $(e_n \cdot e_j)\, g^{jk} = g_{nj} g^{jk} = \delta_n^k$.

From this we conclude that g^{jk} are the components of a contravariant tensor. The tensor g^{jk} is called the contravariant metric tensor. The components g^{jk} of this tensor can be calculated by means of (1.33).

We now multiply the contravariant vector A^m by the metric tensor g_{kr} and contract; we then obtain the covariant vector $g_{kr} A^k$, which will be denoted by A_r. Consequently,

$$A_r = g_{kr} A^k. \tag{1.34}$$

Likewise,

$$A^r = g^{kr} A_k, \tag{1.35}$$

$$g_k^r = g_{kn} g^{nr}. \tag{1.36}$$

The vectors A_r and A^r related by formulas (1.34) and (1.35) are called associated vectors. As seen from formulas (1.34) and (1.35), we can easily calculate the components of either of the vectors A_r or A^r from the components of the other. Hence, A_r and A^r may conveniently be considered as different, respectively covariant and contravariant, components of the same vector A.

The tensor g_k^r is called the mixed metric tensor. It is easy to prove that g_k^r is identical with the Kronecker tensor. Indeed, on the basis of formulas (1.34), (1.35), and (1.36) we have

$$A_k = g_{kr} A^r = g_{kr} g^{mr} A_m = g_k^m A_m,$$

whence

$$A_k = g_k^m A_m.$$

For these relations to be fulfilled for all values of A_k, the components of the mixed metric tensor g_k^m must be chosen as follows:

$$g_k^m = \begin{cases} 1 \text{ when } k = m, \\ 0 \text{ when } k \neq m, \end{cases} \tag{1.37}$$

Consider now the covariant tensor A_{rk}. If the first index r is to be raised, the tensor A_{rk} must be multiplied by g^{mn} and then contracted with respect to the first index, i.e.,

$$A^r_{.k} = g^{rm} A_{mk}.$$

In order that one may know which index has been raised, a dot is inserted in its place. For example, in the equality

$$A_{r.}^{k} = g^{kn} A_{rn}$$

the second index k has been raised. Both indices can be raised by the formula

$$A^{rk} = g^{rm} g^{kn} A_{mn}.$$

These operations can obviously be completely extended to tensors of any rank. All tensors obtained from each other in this way are called associated tensors; their components may also be regarded as the components of the same tensor.

In a rectilinear rectangular system of co-ordinates the square of the distance between a point A of co-ordinates x_m and a point A_1 of co-ordinates $x_m + dx_m$ is given by

$$ds^2 = dx_m^2. \tag{1.38}$$

Since

$$x_m = x_m(x^1,\ x^2,\ x^3),$$

it follows that

$$dx_m = \frac{\partial x_m}{\partial x^n} dx^n.$$

Formula (1.38) may now be put into the form

$$ds^2 = \frac{\partial x_m}{\partial x^n} \frac{\partial x_m}{\partial x^k} dx^n dx^k.$$

By introducing the notation

$$g_{nk} = \frac{\partial x_m}{\partial x^n} \frac{\partial x_m}{\partial x^k} = g_{kn} \tag{1.39}$$

the last formula may be represented as

$$ds^2 = g_{nk} dx^n dx^k. \tag{1.40}$$

We conclude from the quotient law of tensors that g_{nk} is a covariant tensor.

We now determine the values of the contravariant and covariant components of a vector $\boldsymbol{a}$ given at a point P of space. Draw through this point three co-ordinate surfaces

$$x^k = \text{constant}. \tag{1.41}$$

The intersections of these co-ordinate surfaces determine three co-ordinate lines.

We calculate the angles that the directions $\boldsymbol{e}_k$ and $\boldsymbol{e}^k$ make with the axes of the rectangular Cartesian system of co-ordinates x_m. Take an elementary vector $d\boldsymbol{r}_k$ along $\boldsymbol{e}_k$; its length is determined by the formula

$$ds_k = |d\boldsymbol{r}_k| = \sqrt{dx_m^2}.$$

Here the index m is summed from 1 to 3. Taking into account that the co-ordinates x_m along the co-ordinate line x^k depend only on the co-ordinate x^k, the last formula is put into the form

$$ds_k = \sqrt{\left(\frac{\partial x_m}{\partial x^k}\right)^2} dx^k$$

or, by formulas (1.39),

$$ds_k = \sqrt{g_{kk}}\, dx^k.$$

The expression for the cosine of the angle between the direction $\boldsymbol{e}_k$ and the x_m axis then becomes

$$\cos(\boldsymbol{e}_k, x_m) = \frac{dx_m}{ds_k} = \frac{1}{\sqrt{g_{kk}}} \frac{\partial x_m}{\partial x^k}. \tag{1.42}$$

As is known, the formula for the cosine of the angle between the direction $\boldsymbol{e}^k$ (or between the vector grad x^k) and the x_m axis is of the form

$$\cos(\boldsymbol{e}^k, x_m) = \frac{1}{|\operatorname{grad} x^k|} \frac{\partial x^k}{\partial x_m} = \frac{1}{\sqrt{\left(\frac{\partial x^k}{\partial x_j}\right)^2}} \frac{\partial x^k}{\partial x_m}. \tag{1.43}$$

Taking into account that the components of the tensor g_{nk} in a rectangular Cartesian co-ordinate system are equal to δ_n^k, we have, by (1.10),

$$g^{nk} = \delta_i^j \frac{\partial x^n}{\partial x_i} \frac{\partial x^k}{\partial x_j};$$

from this

$$g^{kk} = \left(\frac{\partial x^k}{\partial x_j}\right)^2.$$

On the basis of this formula we find from (1.43)

$$\cos(\boldsymbol{e}^k, x_m) = \frac{1}{\sqrt{g^{kk}}} \frac{\partial x^k}{\partial x_m}. \tag{1.44}$$

Denote the contravariant and covariant components of the vector $\boldsymbol{a}$ by A^k and A_k, and its components in the rectangular Cartesian co-ordinate system by a_m. Further, let a_{e_k} and a_{e^k} denote the projections of the vector $\boldsymbol{a}$, respectively, on $\boldsymbol{e}_k$ and $\boldsymbol{e}^k$. Noting that $\boldsymbol{a} = \boldsymbol{i}_m a_m$ ($\boldsymbol{i}_m$ are the unit vectors of the rectangular Cartesian co-ordinate system), according to the basic formula for the projection of a vector on a given direction we obtain

$$a_{e_k} = a_m \cos(\boldsymbol{e}_k, x_m) = \frac{1}{\sqrt{g_{kk}}} \frac{\partial x_m}{\partial x^k} a_m, \tag{1.45}$$

$$a_{e^k} = a_m \cos(\boldsymbol{e}^k, x_m) = \frac{1}{\sqrt{g^{kk}}} \frac{\partial x^k}{\partial x_m} a_m. \tag{1.46}$$

On the other hand, on the basis of formulas (1.5) and (1.7) we have

$$A^k = a_m \frac{\partial x^k}{\partial x_m}, \quad A_k = a_m \frac{\partial x_m}{\partial x^k}.$$

Substituting these expressions in (1.45) and (1.46), respectively, we obtain, finally,

$$a_{e_k} = \frac{1}{\sqrt{g_{kk}}} A_k, \tag{1.47}$$

$$a_{e^k} = \frac{1}{\sqrt{g^{kk}}} A^k. \tag{1.48}$$

We note once more that the index k is not to be summed in (1.47) and (1.48). If the curvilinear system of co-ordinates is orthogonal, the directions $\boldsymbol{e}_k$ and $\boldsymbol{e}^k$ coincide, and $a_{e_k} = a_{e^k}$; denote these by a_{x^k}.

If the curvilinear system of co-ordinates is orthogonal, then, as is known,

$$g_{kk} = H_k^2, \quad g^{kk} = \frac{1}{H_k^2},$$

where H_k are scale factors.

In this case we obtain from (1.47) and (1.48)

$$a_{x^k} = \frac{1}{H_k} A_k = H_k A^k. \tag{1.49}$$

The a_{x^k} are called the physical projections of the vector $\boldsymbol{a}$.

We now denote a tensor of rank two in rectilinear rectangular co-ordinates x_i by p_{ik}, the physical projections of this tensor in curvilinear orthogonal co-ordinates x^i by $p_{x^i x^k}$, and its contravariant components by A^{ik}; by the formulas (1.10) for the transformation of the components of a tensor we then have

$$A^{ik} = p_{\alpha\beta} \frac{\partial x^i}{\partial x_\alpha} \frac{\partial x^k}{\partial x_\beta}.$$

Taking into account (1.44) and noting that $g^{kk} = H_k^{-2}$, we can write

$$A^{ik} = \frac{1}{H_i H_k} p_{\alpha\beta} \cos(\boldsymbol{e}^i, x_\alpha) \cos(\boldsymbol{e}^k, x_\beta).$$

On the basis of (1.13)

$$p_{x^i x^k} = p_{\alpha\beta} \cos(\boldsymbol{e}^i, x_\alpha) \cos(\boldsymbol{e}^k, x_\beta);$$

then

$$p_{x^i x^k} = H_i H_k A^{ik}.$$

Noting that $A^{ik} = g^{ii} g^{kk} A_{ik} = H_i^{-2} H_k^{-2} A_{ik}$, we have

$$p_{x^i x^k} = H_i H_k A^{ik} = \frac{1}{H_i H_k} A_{ik}. \tag{1.50}$$

Let us determine the angle ϑ between two arbitrary vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ given at the same point. The vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ can be determined by a linear combination of the form

$$A = A^k e_k = A_k e^k,$$
$$B = B^n e_n = B_n e^n.$$

The scalar product of the vectors A and B is

$$A \cdot B = (e_k \cdot e_n) A^k B^n = (e^k \cdot e^n) A_k B_n = (e_k \cdot e^n) A^k B_n = (e^k \cdot e_n) A_k B^n$$

or

$$A \cdot B = g_{kn} A^k B^n = g^{kn} A_k B_n = A^k B_k = A_k B^k. \tag{1.51}$$

On the other hand,

$$A \cdot B = |A| |B| \cos \vartheta, \tag{1.52}$$

where

$$\left.\begin{aligned} |A| &= \sqrt{A \cdot A} = \sqrt{g_{kn} A^k A^n} = \sqrt{g^{kn} A_k A_n} = \sqrt{A_k A^k}, \\ |B| &= \sqrt{g_{kn} B^k B^n} = \sqrt{g^{kn} B_k B_n} = \sqrt{B_k B^k}. \end{aligned}\right\} \tag{1.53}$$

Substituting (1.51) and (1.53) in (1.52), we find

$$\cos \vartheta = \frac{g_{kn} A^k B^n}{\sqrt{g_{kn} A^k A^n} \sqrt{g_{kn} B^k B^n}}. \tag{1.54}$$

From this we obtain a condition for the orthogonality of two vectors A and B:

$$g_{kn} A^k B^n = g^{kn} A_k B_n = A^k B_k = A_k B^k = 0. \tag{1.55}$$

4. DIFFERENTIATION OF BASE VECTORS. THE CHRISTOFFEL SYMBOLS

It follows from (1.26) that

$$\frac{\partial r}{\partial x^n} = e_n \tag{1.56}$$

The base vectors are, in general, functions of position of the point at which they define the co-ordinate trihedral. The variations of the base vectors are characterized by the values of the derivatives $\frac{\partial e_n}{\partial x^k}$. In an Euclidean space the derivative of a vector with respect to a scalar argument is obviously also a vector.

The values of $\frac{\partial e_n}{\partial x^k}$ are represented as the sum of three vectors parallel to the base vectors e_n, i.e.,

$$\frac{\partial e_n}{\partial x^k} = \Gamma^j_{nk} e_j \tag{1.57}$$

The quantities Γ^j_{nk} are called the Christoffel symbols, or they are sometimes referred to as the three-index symbols. If the co-ordinates are Cartesian, then e_n are constant vectors, hence $\Gamma^j_{nk} \equiv 0$, while for a curvilinear co-ordinate system $\Gamma^j_{nk} \neq 0$.

It follows from (1.56) that

$$\frac{\partial e_n}{\partial x^k} = \frac{\partial e_k}{\partial x^n}. \tag{1.58}$$

On the basis of (1.58) we have from formulas (1.57):

$$\Gamma^j_{nk} e_j = \Gamma^j_{kn} e_j,$$

from which it follows that

$$\Gamma^j_{nk} = \Gamma^j_{kn}. \tag{1.59}$$

The Christoffel symbols are expressed in terms of the derivatives of the metric tensor. By multiplying equality (1.57) scalarly by e_m, and taking into account (1.28) we obtain

$$\frac{\partial g_{nm}}{\partial x^k} - e_n \cdot \frac{\partial e_m}{\partial x^k} = \Gamma^j_{nk} g_{jm} \tag{1.60}$$

By interchanging the indices n and k in equality (1.60), and using (1.59), we find

$$\frac{\partial g_{km}}{\partial x^n} - e_k \cdot \frac{\partial e_m}{\partial x^n} = \Gamma^j_{nk} g_{jm} \tag{1.61}$$

By adding (1.60) and (1.61), we have

$$\frac{\partial g_{nm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^n} - \left(e_n \cdot \frac{\partial e_m}{\partial x^k} + e_k \cdot \frac{\partial e_m}{\partial x^n} \right) = 2\Gamma^j_{nk} g_{jm}. \tag{1.62}$$

Calculate the parenthetical expression in this formula. Taking into account relation (1.58), we find

$$\frac{\partial g_{kn}}{\partial x^m} = \frac{\partial}{\partial x^m} (e_n \cdot e_k) = e_n \cdot \frac{\partial e_k}{\partial x^m} + e_k \cdot \frac{\partial e_n}{\partial x^m} = e_n \frac{\partial e_m}{\partial x^k} + e_k \cdot \frac{\partial e_m}{\partial x^n}.$$

Consequently, formula (1.62) becomes

$$2\Gamma^j_{nk} g_{jm} = \frac{\partial g_{nm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^n} - \frac{\partial g_{kn}}{\partial x^m}. \tag{1.63}$$

On multiplying (1.63) by $g^{\alpha m}$, and summing the index m, we obtain, with (1.36) and (1.37),

$$\Gamma^\alpha_{nk} = \frac{1}{2} g^{\alpha m} \left(\frac{\partial g_{nm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^n} - \frac{\partial g_{kn}}{\partial x^m} \right). \tag{1.64}$$

From this it is also seen that the Christoffel symbols are symmetric in the indices n and k.

It can easily be shown that the following equality holds:

$$\frac{\partial e^n}{\partial x^k} = -\Gamma^n_{jk} e^j. \tag{1.65}$$

By differentiating the equality $e^n \cdot e_m = \delta^n_m$, we obtain

$$\frac{\partial e^n}{\partial x^k} \cdot e_m + e^n \cdot \frac{\partial e_m}{\partial x^k} = 0.$$

Taking into account (1.57), we find

$$\frac{\partial e^n}{\partial x^k} \cdot e_m = -(e^n \cdot e_j)\, \Gamma^j_{mk} = -\Gamma^n_{mk}. \tag{1.65'}$$

We now represent $\frac{\partial e^n}{\partial x^k}$ as

$$\frac{\partial e^n}{\partial x^k} = B^n_{jk} e^j.$$

Multiplying both sides of this equality scalarly by e_m gives

$$\frac{\partial e^n}{\partial x^k} \cdot e_m = B^n_{jk} \delta^j_m = B^n_{mk}.$$

On comparing this relation with expression (1.65′), we find

$$B^n_{mk} = -\Gamma^n_{mk}.$$

From this we arrive at formula (1.65).

5. A PARALLEL FIELD OF VECTORS

As we have seen above, the algebraic operations on tensors again lead to tensors, which cannot be said, as we shall see below, about their differentiation. The partial derivatives of the components of a tensor constitute a tensor only in a Cartesian co-ordinate system. In curvilinear co-ordinate systems the situation is more complicated. Here we have to introduce so-called covariant differentiation whose action on a tensor again gives a tensor. The covariant derivative is identical with the ordinary derivative when the tensor is referred to a Cartesian co-ordinate system.

If $f(s)$ is a scalar function (s is the parameter), then $\bar{f} = f$ in the new co-ordinates, and hence

$$\lim_{\Delta s \to 0} \frac{f(s+\Delta s) - f(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\bar{f}(s+\Delta s) - \bar{f}(s)}{\Delta s}$$

or

$$\frac{d\bar{f}}{ds} = \frac{df}{ds},$$

from which it is seen that the derivative of a scalar function with respect to the parameter is again a scalar.

The definition of the covariant derivative of a vector and a tensor will be given in Sec. 6. We first turn our attention to the investigation of a parallel field of vectors.

Let the co-ordinates of an arbitrary point P on the curve under consideration be functions of the parameter s. At each point of this curve construct a vector equal to the vector given at the point P. Thus, we have a parallel field of vectors along the curve. We shall derive equations which this field must satisfy.

Denote the components of the vector field under consideration in the system of co-ordinates x^k by A^k, in the Cartesian system of co-ordinates x_m by a_m. In the Cartesian co-ordinate system the components of parallel vectors are constant along the curve, and hence $\frac{da_m}{ds} = 0$.

By the definition of a vector,

$$a_m = A^k \frac{\partial x_m}{\partial x^k}.$$

By differentiating the last equality with respect to the parameter s, we have

$$\frac{da_m}{ds} = \frac{dA^k}{ds} \frac{\partial x_m}{\partial x^k} + A^k \frac{\partial^2 x_m}{\partial x^n \partial x^k} \frac{dx^n}{ds} = 0. \tag{1.66}$$

On multiplying Eqs. (1.66) by $g^{\alpha\beta} \frac{\partial x_m}{\partial x^\alpha}$, and summing the index m from 1 to 3, we find, from (1.39) and (1.37),

$$\frac{dA^\beta}{ds} + g^{\alpha\beta} \frac{\partial^2 x_m}{\partial x^n \partial x^k} \frac{\partial x_m}{\partial x^\alpha} A^k \frac{dx^n}{ds} = 0. \tag{1.67}$$

By differentiating equalities (1.39) with respect to x^α, we further have

$$\frac{\partial g_{nk}}{\partial x^\alpha} = \frac{\partial^2 x_m}{\partial x^\alpha \partial x^n} \frac{\partial x_m}{\partial x^k} + \frac{\partial x_m}{\partial x^n} \frac{\partial^2 x_m}{\partial x^\alpha \partial x^k}. \tag{1.68}$$

In equalities (1.68) we make twice a cyclic permutation of the indices n, k, α, and subtract (1.68) from the sum of the equalities thus obtained. We find

$$2 \frac{\partial^2 x_m}{\partial x^k \partial x^n} \frac{\partial x_m}{\partial x^\alpha} = \frac{\partial g_{n\alpha}}{\partial x^k} + \frac{\partial g_{\alpha k}}{\partial x^n} - \frac{\partial g_{kn}}{\partial x^\alpha}. \tag{1.69}$$

Substituting (1.69) in (1.67), we have

$$\frac{dA^\beta}{ds} + \Gamma^\beta_{nk} A^k \frac{dx^n}{ds} = 0. \tag{1.70}$$

Thus, the parallel vector field along the given curve must satisfy the differential equations (1.70).

Take any vector at a given point of space and construct vectors parallel to it at all points of space. The components A^β of this parallel vector field are functions of the co-ordinates x^β. If a curve is drawn through any point of this field, the vectors on that curve obviously satisfy Eqs. (1.70). But we now have

$$\frac{dA^\beta}{ds}=\frac{\partial A^\beta}{\partial x^n}\frac{dx^n}{ds},$$

and Eqs. (1.70) become

$$\left(\frac{\partial A^\beta}{\partial x^n}+\Gamma^\beta_{nk}A^k\right)\frac{dx^n}{ds}=0. \tag{1.71}$$

Taking into account that condition (1.71) must be true for all curves issuing from the point P, we find that the parallel vector field satisfies a system of differential equations of the form

$$\frac{\partial A^\beta}{\partial x^n}+\Gamma^\beta_{nk}A^k=0 \tag{1.72}$$

6. THE RIEMANN-CHRISTOFFEL TENSOR. DERIVATIVE OF A VECTOR. THE GAUSS-OSTROGRADSKY FORMULA. THE ε-TENSOR

We now pass on to the determination of new tensors by differentiating given vectors and tensors. Let f be a given scalar function of the co-ordinates of a point x^k. In the new co-ordinates $\bar{x}^m$ related to x^k by formulas (1.1) we then have $\bar{f}=f$. Taking into account the last equality, and using (1.1), we have

$$\frac{\partial \bar{f}}{\partial \bar{x}^m}=\frac{\partial f}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^m}. \tag{1.73}$$

Thus, the derivative of a scalar function with respect to the co-ordinates gives a covariant vector $\frac{\partial f}{\partial x^k}$.

Consider the parallel vector field of an arbitrary contravariant vector A^β along some curve and a covariant vector B_β defined on the same curve. At any point of the given curve the product $B_\beta A^\beta$ is a scalar function of the parameter s, and hence $\frac{d}{ds}(B_\beta A^\beta)$ is also a scalar. On the right-hand side of the equality

$$\frac{d}{ds}(B_\beta A^\beta)=\frac{dB_\beta}{ds}A^\beta+B_\beta\frac{dA^\beta}{ds}$$

we substitute $-\Gamma^\beta_{nk}A^k\frac{dx^n}{ds}$ for $\frac{dA^\beta}{ds}$ from (1.70); then

$$\frac{d}{ds}(B_\beta A^\beta)=\frac{dB_\beta}{ds}A^\beta-\Gamma^\beta_{kn}B_\beta A^k\frac{dx^n}{ds}$$

or

$$\frac{d}{ds}(B_\beta A^\beta)=\left(\frac{dB_\beta}{ds}-\Gamma^k_{\beta n}B_k\frac{dx^n}{ds}\right)A^\beta. \tag{1.74}$$

Taking into account that A^β is an arbitrary contravariant vector of the parallel vector field and that the product of such a vector with the expression inside the parentheses on the right-hand side of (1.74) is a scalar, we conclude from the quotient law of tensors that

$$\frac{dB_\beta}{ds} - \Gamma^k_{\beta n} B_k \frac{dx^n}{ds} \tag{1.75}$$

is a covariant vector.

The covariant vector (1.75) is called the absolute derivative of the covariant vector B_β with respect to the parameter s and is denoted by $\frac{\delta B_\beta}{\delta s}$; consequently,

$$\frac{\delta B_\beta}{\delta s} = \frac{dB_\beta}{ds} - \Gamma^k_{\beta n} B_k \frac{dx^n}{ds}. \tag{1.76}$$

Suppose that in any one co-ordinate system the following equation is satisfied:

$$\frac{dB_\beta}{ds} - \Gamma^k_{\beta n} B_k \frac{dx^n}{ds} = 0; \tag{1.77}$$

it is then satisfied in every other system. In the case of a Cartesian co-ordinate system ($\Gamma^k_{\beta n} \equiv 0$) Eq. (1.77) becomes

$$\frac{dB_\beta}{ds} = 0.$$

It follows that the vector B_β is also a parallel vector field along the curve under consideration. Thus, (1.77) are the equations that the covariant parallel vector field B_β along the given curve must satisfy.

Consider the parallel vector field of an arbitrary covariant vector A_β along a given curve and a contravariant vector B^β defined on the same curve. Proceeding in the same way as in the derivation of formula (1.76), and taking into account that the parallel vector field A_β must satisfy (1.77), we obtain the absolute derivative of the contravariant vector with respect to the parameter s

$$\frac{\delta B^\beta}{\delta s} = \frac{dB^\beta}{ds} + \Gamma^\beta_{kn} B^k \frac{dx^n}{ds}. \tag{1.78}$$

Let B^β be a field of contravariant vectors defined in some space. Take an arbitrary curve passing through any fixed point of this space; from (1.78) and

$$\frac{dB^\beta}{ds} = \frac{\partial B^\beta}{\partial x^n} \frac{dx^n}{ds}$$

we find

$$\frac{\delta B^\beta}{\delta s} = \left(\frac{\partial B^\beta}{\partial x^n} + \Gamma^\beta_{kn} B^k \right) \frac{dx^n}{ds}.$$

Since $\frac{dx^n}{ds}$ is an arbitrary contravariant vector and $\frac{\delta B^\beta}{\delta s}$ is a contravariant vector, it follows from the quotient law that the expression in the parentheses is a mixed tensor of rank two. This tensor is called the covariant derivative of the vector B^β and is denoted by $B^\beta_{,n}$; here the comma before the index n indicates differentiation with respect to x^n. Consequently,

$$B^\beta_{,n} = \frac{\partial B^\beta}{\partial x^n} + \Gamma^\beta_{kn} B^k. \tag{1.79}$$

In a similar way we obtain a covariant tensor of rank two

$$B_{\beta, n} = \frac{\partial B_\beta}{\partial x^n} - \Gamma^k_{\beta n} B_k. \tag{1.80}$$

This tensor is called the covariant derivative of the vector B_β.

We shall now extend these results to the differentiation of a tensor. Consider arbitrary parallel vector fields B^m and C^n defined along some curve. Let A_{mn} be a tensor of rank two defined along the same curve. At each point of this curve $A_{mn}B^mC^n$ gives a scalar; hence, its derivative with respect to s is also a scalar

$$\frac{d}{ds}(A_{mn}B^mC^n) = \frac{dA_{mn}}{ds} B^mC^n + A_{mn}\frac{dB^m}{ds} C^n + A_{mn}B^m\frac{dC^n}{ds}.$$

By eliminating the derivatives of the vectors by means of (1.70), we obtain

$$\frac{d}{ds}(A_{mn}B^mC^n) = \left(\frac{dA_{mn}}{ds} - \Gamma^\alpha_{mt} A_{\alpha n}\frac{dx^t}{ds} - \Gamma^\alpha_{nt} A_{m\alpha}\frac{dx^t}{ds}\right) B^mC^n.$$

Consequently, we conclude from the quotient law that the expression in the parentheses on the right-hand side is a tensor of the same type and rank as A_{mn}. It is called the absolute derivative of the tensor A_{mn} and is denoted by $\frac{\delta A_{mn}}{\delta s}$. Then

$$\frac{\delta A_{mn}}{\delta s} = \frac{dA_{mn}}{ds} - \Gamma^\alpha_{mt} A_{\alpha n}\frac{dx^t}{ds} - \Gamma^\alpha_{nt} A_{m\alpha}\frac{dx^t}{ds}. \tag{1.81}$$

Consider a tensor field in some space. If a curve is taken passing through any point of this space, expression (1.81) along this curve is a tensor.

Noting that

$$\frac{dA_{mn}}{ds} = \frac{\partial A_{mn}}{\partial x^\beta}\frac{dx^\beta}{ds},$$

formula (1.81) may be put into the form

$$\frac{\delta A_{mn}}{\delta s} = \left(\frac{\partial A_{mn}}{\partial x^\beta} - \Gamma^\alpha_{m\beta} A_{\alpha n} - \Gamma^\alpha_{n\beta} A_{m\alpha}\right)\frac{dx^\beta}{ds}. \tag{1.82}$$

Taking into account that (1.82) is a tensor and $\frac{dx^\beta}{ds}$ is an arbitrary contravariant vector, we conclude from the quotient law that the expression in the parentheses is a tensor with one covariant index more than A_{mn}; this tensor is called the covariant derivative of the tensor A_{mn} and is denoted by $A_{mn,\,\beta}$. Here the comma indicates differentiation with respect to x^β; consequently,

$$A_{mn,\beta}=\frac{\partial A_{mn}}{\partial x^\beta}-\Gamma^\alpha_{m\beta}A_{\alpha n}-\Gamma^\alpha_{n\beta}A_{m\alpha}. \tag{1.83}$$

The foregoing method can be used to evaluate the absolute and covariant derivatives of a tensor of any type and of arbitrarily high rank. Thus, if a tensor is specified by contravariant components A^{mn}, then

$$A^{mn}_{,\beta}=\frac{\partial A^{mn}}{\partial x^\beta}+\Gamma^m_{\alpha\beta}A^{\alpha n}+\Gamma^n_{\alpha\beta}A^{m\alpha}. \tag{1.84}$$

Let us evaluate the second covariant derivative of the covariant vector B_r, i.e., with the aid of formula (1.83) we determine the covariant derivative of the tensor $B_{r,s}$:

$$B_{r,st}=\frac{\partial B_{r,s}}{\partial x^t}-\Gamma^m_{rt}B_{m,s}-\Gamma^m_{st}B_{r,\,m}=\frac{\partial^2 B_r}{\partial x^t\,\partial x^s}-\Gamma^m_{rs}\frac{\partial B_m}{\partial x^t}-$$
$$-\Gamma^m_{rt}\frac{\partial B_m}{\partial x^s}-\Gamma^m_{st}\frac{\partial B_r}{\partial x^m}-B_m\left(\frac{\partial}{\partial x^t}\Gamma^m_{rs}-\Gamma^m_{rp}\Gamma^p_{st}-\Gamma^m_{sp}\Gamma^p_{rt}\right).$$

Let us permute the indices s and t in this formula and subtract one expression from the other. By using the symmetry property of the Christoffel symbols, we have

$$B_{r,\,st}-B_{r,\,ts}=R^p_{.rst}B_p, \tag{1.85}$$

where

$$R^p_{.rst}=\frac{\partial}{\partial x^s}\Gamma^p_{rt}-\frac{\partial}{\partial x^t}\Gamma^p_{rs}+\Gamma^m_{rt}\Gamma^p_{ms}-\Gamma^m_{rs}\Gamma^p_{mt}. \tag{1.86}$$

The left-hand side of (1.85) is a tensor and B_p is an arbitrary covariant vector; we conclude from the quotient law that $R^p_{.rst}$ is also a tensor. The tensor $R^p_{.rst}$, which is called the Riemann-Christoffel tensor, consists only of the components of the covariant metric tensor g_{mn} and their derivatives up to the second order.

Let us lower the index p in (1.85), i.e.,

$$R_{prst}=g_{pm}R^m_{.rst}=g_{pm}\frac{\partial}{\partial x^s}\Gamma^m_{rt}-g_{pm}\frac{\partial}{\partial x^t}\Gamma^m_{rs}+\Gamma^m_{rt}\Gamma_{p,\,ms}-\Gamma^m_{rs}\Gamma_{p,\,mt},$$

where

$$\Gamma_{p,\,ms}=\frac{1}{2}\left(\frac{\partial g_{mp}}{\partial x^s}+\frac{\partial g_{ps}}{\partial x^m}-\frac{\partial g_{ms}}{\partial x^p}\right),$$
$$\Gamma_{m,\,ps}=\frac{1}{2}\left(\frac{\partial g_{mp}}{\partial x^s}+\frac{\partial g_{ms}}{\partial x^p}-\frac{\partial g_{ps}}{\partial x^m}\right);$$

from this

$$\frac{\partial g_{mp}}{\partial x^s}=\Gamma_{p,\,ms}+\Gamma_{m,\,ps}.$$

Noting that

$$g_{pm}\frac{\partial}{\partial x^s}\Gamma^m_{rt}=\frac{\partial}{\partial x^s}(g_{pm}\Gamma^m_{rt})-\Gamma^m_{rt}\frac{\partial g_{pm}}{\partial x^s}=$$

$$=\frac{\partial}{\partial x^s}\Gamma_{p,\,rt}-\Gamma^m_{rt}(\Gamma_{p,\,ms}+\Gamma_{m,\,ps}),$$

we find

$$R_{prst}=\frac{\partial}{\partial x^s}\Gamma_{p,\,rt}-\frac{\partial}{\partial x^t}\Gamma_{p,\,rs}+\Gamma^m_{rs}\Gamma_{m,\,pt}-\Gamma^m_{rt}\Gamma_{m,\,ps}.$$

Inserting the expressions for $\Gamma_{p,\,ms}$ under the derivative signs in the last equality, and using formula (1.64), this becomes

$$R_{prst}=\frac{1}{2}\left(\frac{\partial^2 g_{pt}}{\partial x^s\,\partial x^r}+\frac{\partial^2 g_{rs}}{\partial x^t\,\partial x^p}-\frac{\partial^2 g_{ps}}{\partial x^t\,\partial x^r}-\frac{\partial^2 g_{rt}}{\partial x^s\,\partial x^p}\right)+$$
$$+g^{mn}(\Gamma_{m,\,rs}\Gamma_{n,\,pt}-\Gamma_{m,\,rt}\Gamma_{n,\,ps}).\qquad(1.87)$$

The properties given below follow directly from the last formulas:

$$\begin{aligned}&R_{prst}=-R_{rpst},\quad R_{ppst}=0,\\&R_{prst}=-R_{prts},\quad R_{prtt}=0.\\&R_{prst}=R_{stpr}.\end{aligned}\qquad(1.88)$$

The equalities in the first two lines express the antisymmetry of the tensor R_{prst} with respect to each pair of indices p, r and s, t. Taking into account properties (1.88), after calculation we find that, of 81 components of the Riemann-Christoffel tensor, there are only six independent components, namely R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{2123}, R_{3132}.

It is known that a Cartesian co-ordinate system can be introduced into the whole Euclidean space. Since the components of the metric tensor are constant in the Cartesian co-ordinate system, and hence the Christoffel symbols are zero, from formulas (1.87) we have

$$R_{prst}=0.\qquad(1.89)$$

Thus, conditions (1.89) are necessary conditions for a space to be Euclidean.

The converse may also be proved. If the Riemann-Christoffel tensor vanishes at all points of space, co-ordinates x^1, x^2, x^3 may be chosen in this space such that the quadratic form will become $ds^2=g_{ik}dx^ix^k$, with constant coefficients. The constancy of these coefficients indicates that the space is Euclidean. Consequently, the condition that the Riemann-Christoffel tensor is zero provides a sufficient condition for a space to be Euclidean.

It has been shown above that in a three-dimensional space the Riemann-Christoffel tensor has only six independent components. Consequently, conditions (1.89) may be replaced by six independent conditions of the form

$$R_{prst}=0\quad (prst=1212,\ 1313,\ 2323,\ 1213,\ 2123,\ 3132). \tag{1.90}$$

Thus, conditions (1.90) are necessary and sufficient conditions for a space to be Euclidean.

Let us derive the Gauss-Ostrogradsky formula in a curvilinear co-ordinate system.

As is well known, the formula for the transformation of a volume integral into a surface integral in a rectangular co-ordinate system, i.e., the Gauss-Ostrogradsky formula, is of the form

$$\int\limits_{\tau}\frac{\partial a_k}{\partial x^k}\,d\tau=\int\limits_{\omega}a_kl_k\,d\omega, \tag{1.91}$$

where ω is a closed surface bounding the volume τ, l_k are the direction cosines of the outward normal to the surface ω.

Take now a curvilinear system of co-ordinates x^k and let A^k denote a contravariant vector defining the vector a_k in the x^k co-ordinate system. Remembering that A^k_k is a scalar, we have

$$A^k_{,k}=\frac{\partial a_k}{\partial x^k}. \tag{1.92}$$

Denote by n_k the covariant components of the unit outward normal vector whose components in the x_k co-ordinate system are l_k. On the other hand,

$$A_kn_k=a_kl_k. \tag{1.93}$$

Consequently from (1.91), with (1.92) and (1.93), we obtain

$$\int\limits_{\tau}A^k_{,k}\,d\tau=\int\limits_{\omega}A^kn_k\,d\omega. \tag{1.94}$$

In conclusion we consider the ε-tensor. Let the components of an object e_{rst} be altered in sign, but not in absolute value when any two indices are interchanged. Consequently, the components of the symbol e_{rst} can obviously have only the following values: 0, when any two of the indices are equal; +1, when *rst* is an even permutation of the numbers 1, 2, 3; —1, when *rst* is an odd permutation of the numbers 1, 2, 3.

Consider the determinant $|a^r_s|$. Here the upper index denotes the row and the lower index the column. If the determinant is expanded in full by columns, it reads

$$|a^r_s|=\pm a^i_1a^j_2a^k_3.$$

Here the summation is carried out with respect to the indices i, j, k, which form permutations of the numbers 1, 2, 3, and the plus or minus sign is given accordingly as the permutation of these numbers is even or odd. By the definition of the e-symbols, the determinant equals

$$|a^r_s| = e_{ijk}a^i_1 a^j_2 a^k_3.$$

Consider the sum

$$e_{ijk}a^i_\alpha a^j_\beta a^k_\gamma \quad (i,\ j,\ k,\ \alpha,\ \beta,\ \gamma = 1,\ 2,\ 3).$$

Taking into account that the indices i, j, k are summation indices, we have

$$e_{ijk}a^i_\alpha a^j_\beta a^k_\gamma = -e_{kji}a^i_\alpha a^j_\beta a^k_\gamma = -e_{ijk}a^i_\gamma a^j_\beta a^k_\alpha.$$

Thus, the interchange of the indices α and γ alters the sign. The same result holds for any other two of the indices. Consequently, the sum under consideration is antisymmetric in the indices α, β, γ, i.e.,

$$e_{ijk}a^i_\alpha a^j_\beta a^k_\gamma = |a^r_s|\, e_{\alpha\beta\gamma}. \tag{1.95}$$

By putting $a^i_j = \frac{\partial x^i}{\partial \bar{x}^j}$ in formula (1.95), and noting that the Jacobian $\left|\frac{\partial x^i}{\partial \bar{x}^j}\right| \neq 0$, we obtain

$$\bar{e}_{ijk} = \left|\frac{\partial x^r}{\partial \bar{x}^s}\right|^{-1} e_{\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j}\frac{\partial x^\gamma}{\partial \bar{x}^k}. \tag{1.96}$$

Introduce a new co-ordinate system $y^i = y^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ and set up an expression for $e_{ijk}\left|\frac{\partial x^r}{\partial y^s}\right|$. On the basis of formula (1.96) we obtain

$$e_{ijk}\left|\frac{\partial x^r}{\partial y^s}\right| = \left|\frac{\partial \bar{x}^r}{\partial y^s}\right|^{-1}\bar{e}_{\alpha\beta\gamma}\frac{\partial \bar{x}^\alpha}{\partial y^i}\frac{\partial \bar{x}^\beta}{\partial y^j}\frac{\partial \bar{x}^\gamma}{\partial y^k}\left|\frac{\partial x^r}{\partial \bar{x}^s}\right|\left|\frac{\partial \bar{x}^r}{\partial y^s}\right|$$

or, after cancelling out $\left|\frac{\partial x^r}{\partial y^s}\right| \neq 0$, we have

$$e_{ijk}\left|\frac{\partial x^r}{\partial y^s}\right| = \bar{e}_{\alpha\beta\gamma}\left|\frac{\partial x^r}{\partial \bar{x}^s}\right|\frac{\partial \bar{x}^\alpha}{\partial y^i}\frac{\partial \bar{x}^\beta}{\partial y^j}\frac{\partial \bar{x}^\gamma}{\partial y^k}.$$

From this and from the definition of a tensor the quantities $e_{ijk}\left|\frac{\partial x^r}{\partial \bar{x}^s}\right|$ represent a tensor; it is denoted by ε_{ijk}.

Introduce now three non-coplanar vectors, A^i, B^j, C^k. From the definition of the triple scalar product it follows that

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = A^i e_i\,(B^j e_j \times C^k e_k) = \left|\frac{\partial x^i}{\partial \bar{x}^j}\right| e_{ijk}A^iB^jC^k = \varepsilon_{ijk}A^iB^jC^k. \tag{1.97}$$

CHAPTER II

Theory of stress

Initially Cauchy and Navier regarded a solid as a system of material particles. Each pair of particles were assumed to be interconnected by forces of interaction directed along a straight line joining them and linearly dependent on the distance between the particles. With the level at which physics was at the beginning of the nineteenth century, it was impossible to describe the elastic properties of real bodies in this way. At present there are rigorous physical theories which enable one to determine the elastic properties of crystals of different structure proceeding from the consideration of the forces of interaction between the atoms in a crystal lattice. An easier way followed by the modern theory of elasticity is to consider the distribution of the substance of a body to be continuous throughout its volume; this allows the displacements of particles to be assumed as continuous functions of co-ordinates.

To calculate the force of interaction between particles situated on one side of an arbitrary element, imagined to be isolated inside the body, and particles situated on the other side of this element, it was found advantageous to introduce the concept of the averaged force of interaction between them.

The error resulting from the above abstraction may be appreciable in the case of determining the relative displacements of points that are originally spaced apart at distances comparable with the distances between particles, and in determining the force acting on an element of comparable size with the square of the distance between particles.

In solving practical problems of the deformation of a solid this abstraction introduces no serious errors, a fact which justifies the replacement of a solid by a continuous medium.

7. TYPES OF EXTERNAL FORCES

Two types of external forces acting on a body are distinguished.

1. Surface forces are those which arise at points of the body surface.

Let an element $d\omega$ of the body surface be acted on by a force $\boldsymbol{dP}$, then the vector

$$\frac{d\boldsymbol{P}}{d\omega}=\boldsymbol{T}_n \tag{2.1}$$

represents the force per unit surface area at the point M (Fig. 1). It is called the intensity of surface force and its dimension is force/length².

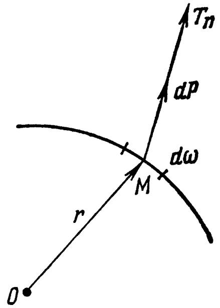

Fig. 1

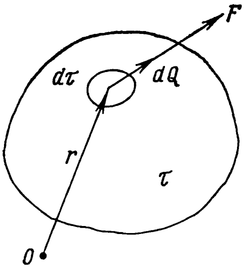

Fig. 2

The resultant vector and the resultant moment of the surface forces applied to the entire surface ω are, by definition,

$$\boldsymbol{V}=\int_{\omega}\boldsymbol{T}_n\,d\omega, \tag{2.2}$$

$$\boldsymbol{L}=\int_{\omega}(\boldsymbol{r}\times\boldsymbol{T}_n)\,d\omega. \tag{2.3}$$

Here $\int_{\omega}$ is the surface integral, $\boldsymbol{r}$ is the radius vector of the point of application of the force with reference to an arbitrarily chosen origin of co-ordinates.

Examples of surface forces are the pressure of liquids or solids that are in contact with a given body, the pressure of light, etc.

2. Mass forces are those which act on an element of mass of a body (Fig. 2). Let the mass $dm=\rho d\tau$ enclosed in an element of volume $d\tau$ of the body be acted on by a force dQ. The vector $F=\frac{dQ}{dm}$ then represents the force per unit mass at the given point.

Volume forces F^* are also considered, which are defined by the formula

$$F^*=\frac{dQ}{d\tau}=\rho\,\frac{dQ}{dm}=\rho F \tag{2.4}$$

The resultant vector and the resultant moment of the volume forces applied to the entire volume τ are

$$V = \int_{\tau} F \rho \, d\tau, \tag{2.5}$$

$$L = \int_{\tau} (r \times F) \rho \, d\tau. \tag{2.6}$$

Here $\int_{\tau}$ is the volume integral.

A typical example of mass forces is provided by gravitational forces. If the x_3 axis is directed vertically downward, the gravity force per unit volume is

$$F^* = i_3 \rho g. \tag{2.7}$$

8. THE METHOD OF SECTIONS. THE STRESS VECTOR

The positions of particles in an undeformed body correspond to its state of thermal equilibrium. If a certain volume is isolated from this body, all forces exerted on it by other parts are balanced. Under the action of external forces, however, the positions of particles in the body change, i.e., the body deforms, with the result that internal forces arise. To determine the latter, use is made of the well-known method of sections. Suppose we have a deformable body which is in equilibrium under the action of external forces. Imagine it to be cut by a surface *mm* into two parts. By removing one part, we replace its action on the remaining part by internal forces distributed over the surface of the section; these are the bonding forces between particles of the body situated on both sides of the section (Fig. 3). The forces acting at the points of the surface of the section may now be classified as external surface forces. For equilibrium of the remain-

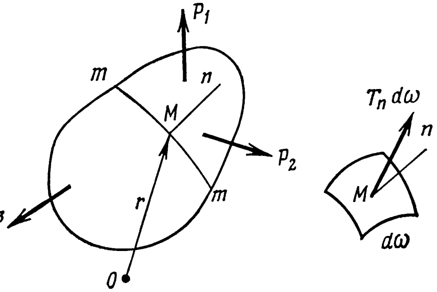

Fig. 3

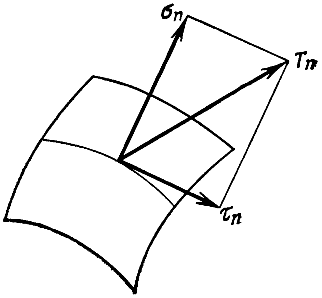

Fig. 4

ing part, these forces must be chosen so that, together with the prescribed forces acting on the part of the body under consideration, they will constitute a balanced system of forces. Denote by $\Delta \boldsymbol{V}$ and $\Delta \boldsymbol{L}$, respectively, the resultant vector and the resultant moment of the forces distributed over a surface element $\Delta\omega$ of the section *mm* with normal $\boldsymbol{n}$ at the point M. The direction of the normal $\boldsymbol{n}$ to the surface element $\Delta\omega$ is considered positive if it is directed from the remaining to the removed part.

Assuming that in the model of a continuous medium considered by us there is only central action between its particles, we have

$$\lim_{\Delta\omega \to 0} \frac{\Delta \boldsymbol{L}}{\Delta\omega} = 0, \quad \lim_{\Delta\omega \to 0} \frac{\Delta \boldsymbol{V}}{\Delta\omega} = \boldsymbol{T}_n.$$

The vector $\boldsymbol{T}_n$ is called the stress vector on the surface element with normal $\boldsymbol{n}$ at the point M.

In considering the model of a medium introduced by W. Voigt in 1887 it is assumed that, in addition to the ordinary central action, there is also rotational action between its particles. Then, besides the stress vector $\boldsymbol{T}_n$, there is also a couple-stress vector $\boldsymbol{M}_n$ equal to

$$\boldsymbol{M}_n = \lim_{\Delta\omega \to 0} \frac{\Delta \boldsymbol{L}}{\Delta\omega} \neq 0.$$

The latter model will not be discussed here.

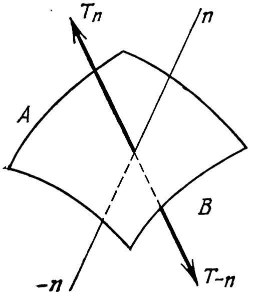

Fig. 5

The dimension of the stress vector, as follows from its definition, is force/length^2.

The stress vector $\boldsymbol{T}_n$ can be resolved into two components:

(1) The normal component directed along the normal $\boldsymbol{n}$ is called the normal stress and denoted by σ_n.

(2) The tangential component directed along the tangent to the curve of intersection of the plane passing through $\boldsymbol{T}_n$ and $\boldsymbol{n}$ and the surface of the section is called the shearing stress and denoted by τ_n (Fig. 4).

The normal stress is commonly considered positive if its sense coincides with the sense of the outward normal to the surface of the section at a given point. Otherwise negative.

If the direction of the stress vector $\boldsymbol{T}_n$ coincides with the normal to the surface of the section at a given point, then

$$\boldsymbol{T}_n = \sigma_n \text{ and } \tau_n = 0.$$

In this case the normal stress is called the principal normal stress, and the area on which this stress is acting is called the principal area at a given point.

The stress referred to the surface element $d\omega$ with normal $\boldsymbol{n}$ in the undeformed state is called the engineering stress, and the stress referred to the surface element $d\omega'$ with normal $\boldsymbol{n}'$ in the deformed state is called the true stress.

Suppose that the stress vector $\boldsymbol{T}_n$ characterizes the action of a body A on a body B transmitted through a surface element of the section with normal $\boldsymbol{n}$, and the stress $\boldsymbol{T}_{-n}$ characterizes the action of the body B on the body A transmitted through the same surface element (Fig. 5).

On the basis of Newton's third law we have the equality

$$\boldsymbol{T}_{-n} = -\boldsymbol{T}_n. \tag{2.8}$$

9. THE STRESS TENSOR

We choose some point P in a body and draw, through it, the co-ordinate lines of an arbitrary curvilinear system of co-ordinates x^k.

Consider, at the point P, a tetrahedron imagined to be isolated from the undeformed body by three co-ordinate surfaces defined by the covariant base vectors $\boldsymbol{e}_k$ and a surface the outward normal to which is a certain direction $\boldsymbol{n}$ ($\boldsymbol{n}$ is the unit vector) passing through the same point P (Fig. 6).

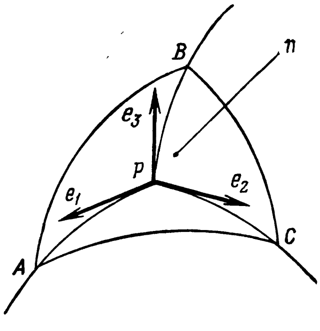

Fig. 6

Consider the motion of the tetrahedron. Denote by $d\omega_1$, $d\omega_2$, $d\omega_3$, and $d\omega$, respectively, the surface areas CPB, APC, APB, and ABC. These surfaces, whose normals are, respectively, the vectors of the reciprocal base $\boldsymbol{e}^k$ and the unit normal $\boldsymbol{n}$, are acted on by the forces $-\boldsymbol{T}_k\, d\omega_k$ ($k = 1, 2, 3$) and $\boldsymbol{T}_n\, d\omega$, where $\boldsymbol{T}_k$ and $\boldsymbol{T}_n$ are the stress vectors on the co-ordinate areas with normals $\boldsymbol{e}^k$ and $\boldsymbol{n}$. Besides, the volume force of the isolated element is $\rho\,(\boldsymbol{F} - \boldsymbol{W})\,d\tau$ (here $\boldsymbol{W}$ is the acceleration, ρ is the density of the material of the undeformed medium).

On the basis of D'Alembert's principle the equation of motion of the tetrahedron as a rigid body is of the form

$$\boldsymbol{T}_n\, d\omega - \boldsymbol{T}_k\, d\omega_k - \rho\,(\boldsymbol{F} - \boldsymbol{W})\,d\tau = 0. \tag{2.9}$$

Here the index k in the second term is summed from 1 to 3. Since the sum of the vectors of the areas for the tetrahedron is zero, we have

$$\boldsymbol{n}\, d\omega = \frac{d\omega_k}{|\boldsymbol{e}^k|}\, \boldsymbol{e}^k. \tag{2.10}$$

Noting that $\boldsymbol{e}^k \cdot \boldsymbol{e}^k = g^{kk}$ and $\boldsymbol{n} = n_k \boldsymbol{e}^k$ (n_k are the covariant components of the unit vector), instead of (2.10) we have

$$n_k \, d\omega \boldsymbol{e}^k = \frac{d\omega_k}{\sqrt{g^{kk}}} \boldsymbol{e}^k,$$

from which

$$d\omega_k = \sqrt{g^{kk}} \, n_k \, d\omega.$$

Substituting this in (2.9), we obtain

$$\boldsymbol{T}_n - \boldsymbol{T}_k \sqrt{g^{kk}} \, n_k - \rho (\boldsymbol{F} - \boldsymbol{W}) \frac{d\tau}{d\omega} = 0.$$

Let the distance from the point P to the surface ABC tend to zero while the direction $\boldsymbol{n}$ is kept constant. Taking into account that $\frac{d\tau}{d\omega} \to 0$, from the last equation we find

$$\boldsymbol{T}_n = \sqrt{g^{kk}} \, \boldsymbol{T}_k n_k. \tag{2.11}$$

The stress vector $\boldsymbol{T}_k$ may be represented by three components referred to the vectors of the covariant base $\boldsymbol{e}_m$, i.e.,

$$\sqrt{g^{kk}} \, \boldsymbol{T}_k = \sigma^{km} \boldsymbol{e}_m. \tag{2.12}$$

Substituting (2.12) in (2.11), we obtain

$$\boldsymbol{T}_n = \sigma^{km} n_k \boldsymbol{e}_m. \tag{2.13}$$

Noting that $\boldsymbol{T}_n = T_n^m \boldsymbol{e}_m$, where T_n^m are the contravariant components of the stress vector, we find

$$T_n^m = \sigma^{km} n_k. \tag{2.14}$$

In (2.13) a double summation is carried out with respect to the indices k and m, and in (2.14) a single summation is carried out with respect to the index k.

Formula (2.14) determines the contravariant components of the stress vector on the area specified by the normal $\boldsymbol{n}$; hence, we conclude from the quotient law of tensors that the σ^{km} constitute the contravariant components of a tensor of rank two. The tensor σ^{km} is called the contravariant stress tensor.

10. EQUATIONS OF MOTION AND EQUILIBRIUM IN TERMS OF THE COMPONENTS OF THE STRESS TENSOR

It is known that in order to set up the equations of motion of an absolutely rigid body it is necessary and sufficient to equate to zero the resultant vector and the resultant moment of the external forces acting on it and the inertia forces.

In order to set up the equations of motion of a deformable body it is necessary and sufficient to equate to zero the resultant vector and the resultant moment of the external and inertia forces applied to each part of the body that can be imagined to be isolated from it.

Equating the resultant vector and the resultant moment of the forces mentioned above to zero imposes certain conditions (which we proceed to derive) on the variation of the components of the stress tensor in passing from one to another point of the body In the following discussion it will be assumed that the components of the stress tensor are continuous and have continuous partial derivatives at all points of the body. Imagine that an arbitrary volume τ bounded by a reasonably smooth surface ω is cut out inside the body. The resultant vector and the resultant moment of the volume forces $\rho \boldsymbol{F}\, d\tau$ acting on a volume element $d\tau$ isolated from the volume τ, the inertia forces $-\rho \boldsymbol{W} d\tau$ applied to this volume in the case of dynamic loading, and the surface forces $\boldsymbol{T}_n\, d\omega$ acting on the element $d\omega$ must be equal to zero, i.e.,

$$\int_\tau \rho\,(\boldsymbol{F}-\boldsymbol{W})\, d\tau + \int_\omega \boldsymbol{T}_n\, d\omega = 0, \tag{2.15}$$

$$\int_\tau (\boldsymbol{r} \times \rho\,(\boldsymbol{F}-\boldsymbol{W}))\, d\tau + \int_\omega (\boldsymbol{r} \times \boldsymbol{T}_n)\, d\omega = 0. \tag{2.16}$$

Since the component of the force $\rho\,(\boldsymbol{F} - \boldsymbol{W})\, d\tau$ in the direction of the unit vector $\boldsymbol{\nu}$ is equal to $\boldsymbol{\nu}\rho\,(\boldsymbol{F} - \boldsymbol{W})\, d\tau$ and the component of the force $\boldsymbol{T}_n\, d\omega$ in the same direction is equal to $\boldsymbol{\nu} \boldsymbol{T}_n\, d\omega$, instead of (2.15) we may write

$$\int_\tau \rho\,(F^m - W^m)\, \nu_m\, d\tau + \int_\omega T_n^m \nu_m\, d\omega = 0. \tag{2.17}$$

Taking into account (2.14) and the Gauss-Ostrogradsky formula (1.94), the surface integral in (2.17) may be put into the form

$$\int_\omega T_n^m \nu_m\, d\omega = \int_\omega (\sigma^{km}\nu_m)\, n_k\, d\omega = \int_\tau (\sigma^{km}\nu_m)_{,k}\, d\tau = \int_\tau \sigma^{km}_{,k}\nu_m\, d\tau^{*}. \tag{2.18}$$

Substituting this in (2.17), we obtain

$$\int_\tau (\rho\,(F^m - W^m) + \sigma^{km}_{,k})\, \nu_m\, d\tau = 0.$$

From this, because of the continuity of the integrand and the arbitrariness of the vector ν_m and the volume τ, it follows that the inte-

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[*] For a parallel vector field in the region ν_m

$$\nu_{m,k} = \frac{\partial \nu_m}{\partial x^k} - \Gamma^\beta_{mk}\nu_\beta = 0.$$

grand must vanish at each point of the body

$$\sigma^{km}_{,k} + \rho F^m = \rho W^m. \tag{2.19}$$

If the body is in equilibrium, the acceleration of the element $d\tau$ is zero and the equation takes the form

$$\sigma^{km}_{,k} + \rho F^m = 0. \tag{2.20}$$

Here $\sigma^{km}_{,k}$ is the covariant derivative of the stress tensor σ^{km}; the index k is summed from 1 to 3.

Equations (2.19) and (2.20) connecting the variation of the components of the stress tensor with the mass forces at any point inside the body are termed, respectively, the equations of motion and the equations of equilibrium of a deformable body in contravariant form. These equations, which involve nine components of the stress tensor, are non-homogeneous partial differential equations of the first order. In the absence of body forces these equations become homogeneous.

Since the component of the moment $\boldsymbol{r} \times \rho\,(\boldsymbol{F} - \boldsymbol{W})\,d\tau$ in the direction of the unit vector $\boldsymbol{\nu}$ is equal to $\boldsymbol{\nu}\cdot\boldsymbol{r}\times\rho\,(\boldsymbol{F}-\boldsymbol{W})d\tau$, the component of the moment $\boldsymbol{r} \times \boldsymbol{T}_n\,d\omega$ in the same direction is equal to $\boldsymbol{\nu}\cdot\boldsymbol{r}\times \boldsymbol{T}_n\,d\omega$, and the radius vector $\boldsymbol{r}$ of the point may be represented as $\boldsymbol{r} = l^j\boldsymbol{e}_j$, instead of (2.16) we write, using formula (1.97),

$$\int_\tau \varepsilon_{ijk}\rho\,(F^i - W^i)\,l^j\nu^k\,d\tau + \int_\omega \varepsilon_{ijk}T^i_n l^j\nu^k\,d\omega = 0. \tag{2.20'}$$

Taking into account formula (2.13) and the Gauss-Ostrogradsky formula (1.94), the surface integral in Eq. (2.20') may be put into the form

$$\int_\omega \varepsilon_{ijk}l^j\nu^k\sigma^{mi}n_m\,d\omega = \int_\tau (\varepsilon_{ijk}l^j\nu^k\sigma^{mi})_{,m}\,d\tau.$$

Since $\varepsilon_{ijk,\,m} = 0$* and $\nu^k_{,m} = 0$, Eq. (2.20') is written as

$$\int_\tau \varepsilon_{ijk}\nu^k\,[\rho\,(F^i - W^i)\,l^j + (\sigma^{mi}l^j)_{,m}]\,d\tau = 0.$$

Since $\sigma^{mi}_{,m} + \rho F^i = \rho W^i$, it follows that

$$\int_\tau \varepsilon_{ijk}\nu^k\sigma^{mi}l^j_{,m}\,d\tau = 0.$$

---

* Since ε_{ijk} is a constant in a Cartesian system of co-ordinates, it follows that $\varepsilon_{ijk,m} = 0$ in this co-ordinate system, and this is also true for every other co-ordinate system.

By virtue of the formula $l^{\alpha}_{,j} = \delta^{\alpha}_{j}$*, the arbitrariness of the volume τ, and the continuity of the integrand we have

$$\varepsilon_{ijk}\sigma^{ji}\nu^{k} = 0.$$

Noting that $\varepsilon_{ijk} = -\varepsilon_{jik}$, the last relation may be represented as

$$\frac{1}{2}\varepsilon_{ijk}\left(\sigma^{ji} - \sigma^{ij}\right)\nu^{k} = 0.$$

By expanding this expression, we have

$$(\sigma^{23} - \sigma^{32})\,\nu^{1} + (\sigma^{31} - \sigma^{13})\,\nu^{2} + (\sigma^{12} - \sigma^{21})\,\nu^{3} = 0.$$

Since the direction $\boldsymbol{\nu}$ is arbitrary, we conclude that

$$\sigma^{km} = \sigma^{mk}. \tag{2.21}$$

The symmetry of the stress tensor is thus proved. Consequently, the stress tensor defining the state of stress at a given point is determined by six independent components.

11. SURFACE CONDITIONS

In the preceding section it has been stated that the necessary and sufficient condition for the equilibrium of a deformable body is that of zero resultant vector and zero resultant moment of the forces applied to each part of the body that can be imagined to be isolated from it. This must also be true for parts of the body having a surface coinciding with the body surface. Assume that the components of the stress tensor are continuous up to the boundary.

The conditions for the equilibrium of an infinitesimal tetrahedron (see Fig. 6), when the surface ABC coincides with the surface of the body, give a relationship between the stress tensor and the external forces. This relationship is of the form of (2.13) or (2.14), with the difference that n in these formulas is the outward normal to the surface of the body at a given point. These conditions are called the surface or boundary conditions.

Thus, from the necessary and sufficient condition of zero resultant vector and zero resultant moment of the forces applied to each part of the body, including parts of the body having a surface coinciding with the body surface, it follows that six components of the stress tensor must satisfy, inside the body, three differential equations (2.19) in the case of dynamic loading or (2.20) in the case of static loading, and three surface conditions (2.14).

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* This formula follows from the relation

$$e_j = \frac{\partial r}{\partial x^j} = \frac{\partial}{\partial x^j}\left(l^{\alpha} e_{\alpha}\right) = \frac{\partial l^{\alpha}}{\partial x^j} e_{\alpha} + l^{\alpha}\frac{\partial e_{\alpha}}{\partial x^j} = \left(\frac{\partial l^{\alpha}}{\partial x^j} + \Gamma^{\alpha}_{\beta j} l^{\beta}\right) e_{\alpha} = l^{\alpha}_{,j} e_{\alpha}$$

It should be noted that six components of the stress tensor are not determined uniquely from the system of three differential equations. Each solution of the infinite number of solutions of this system that satisfies three boundary conditions corresponds to some statically possible state of stress.

Consequently, under the action of applied external forces there may be an infinite number of statically possible states of stress. Thus, the problem of finding the state of stress in a body is statically indeterminate.

Below (Chap. V) it will be shown how the actual state of stress can be determined from the infinite number of statically possible states of stress.

12. EQUATIONS OF MOTION AND EQUILIBRIUM REFERRED TO A CARTESIAN CO-ORDINATE SYSTEM

Let x_k be the axes of a rectangular Cartesian co-ordinate system drawn through some point of a stressed body. The covariant and contravariant components of the stress vector and the stress tensor are then identical and, by formulas (1.55) and (1.56), equal to the physical components. Formulas (2.14) assume the form

$$T_{nk} = \sigma_{mk} n_m. \tag{2.22}$$

Here T_{nk} are the components of the stress vector T_n acting on a plane passing through the given point of the body, the outward normal to which makes angles (n, x_n) with the co-ordinate axes; σ_{mk} are the components of the affine orthogonal stress tensor, σ_{kk} (not to be summed) being the stresses normal to the co-ordinate planes; σ_{mk} $(m \neq k)$ are the shearing stresses.

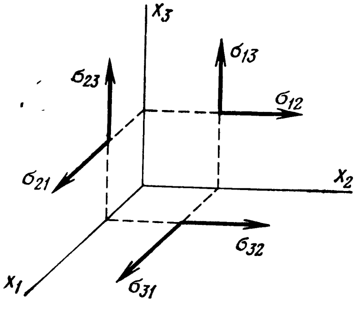

Fig. 7

The symmetry of the stress tensor expresses the law of paired shearing stresses: at every point of a body the shearing stresses on two planes at right angles to each other are perpendicular to the line of intersection of the planes, equal in magnitude and directed either both towards the line of intersection or both away from it (Fig. 7 .

Rotate the axes ox_k about the origin; we then have, by (1.13),

$$\sigma'_{ri} = \sigma_{mk}\alpha_{rm}\alpha_{ik} \tag{2.23}$$

(α_{rm} is the cosine of the angle between the x_r and x_m axes).

Noting that $\alpha_{rs}^2 = 1$, $\alpha_{rs}\alpha_{is} = 0$ $(r \neq i)$, where the index s is summed, from (2.23) we obtain

$$\sigma'_{rr} = \sigma_{rr}.$$

Consequently, the sum of the normal stress components acting on three mutually perpendicular planes is independent of their orientation at the given point

In a rectangular Cartesian co-ordinate system, owing to the fact that the Christoffel symbols vanish and the covariant components of the stress vector and the stress tensor are identical with the physical components, the equations of motion (2.19) and the equations of equilibrium (2.20) become, respectively,

$$\frac{\partial \sigma_{mk}}{\partial x_m} + \rho F_k = \rho W_k, \tag{2.24}$$

$$\frac{\partial \sigma_{mk}}{\partial x_m} + \rho F_k = 0. \tag{2.25}$$

These equations may also be written as follows: in the case of motion

$$\operatorname{div} \boldsymbol{T}_k + \rho F_k = \rho W_k, \tag{2.26}$$

and in the case of equilibrium

$$\operatorname{div} \boldsymbol{T}_k + \rho F_k = 0. \tag{2.27}$$

Here $\boldsymbol{T}_k = \boldsymbol{i}_r \sigma_{kr}$ is the stress vector on the co-ordinate plane $x_k =$ = constant.

13. EQUATIONS OF MOTION AND EQUILIBRIUM REFERRED TO CYLINDRICAL AND SPHERICAL CO-ORDINATES

It is often found convenient to use the equations of motion and equilibrium in a cylindrical and a spherical co-ordinate system.

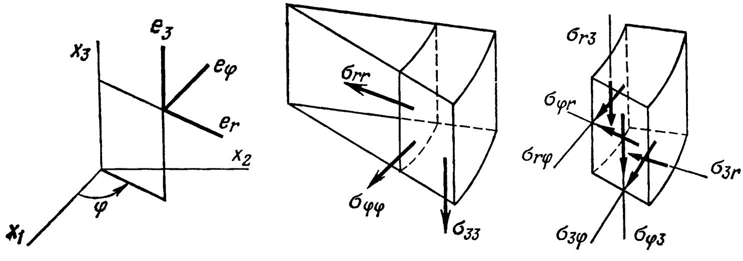

Fig. 8

The physical projections of the force $\rho \boldsymbol{F}$ and the acceleration $\boldsymbol{W}$ in a cylindtical co-ordinate system are denoted, respectively, by ρF_r, ρF_φ, ρF_3 and W_r, W_φ, W_3, and the physical projections of the stress tensor in the same co-ordinate system are denoted by σ_{rr}, $\sigma_{\varphi\varphi}$, σ_{33}, $\sigma_{r\varphi}$, σ_{r3}, $\sigma_{\varphi 3}$ (Fig. 8).

By using formulas (1.49) and (1.50), and taking into account that the components of the covariant metric tensor in a cylindrical co-ordinate system are

$$g_{11}=H_1^2=1,\ g_{22}=H_2^2=r^2,\ g_{33}=H_3^2=1,\ g_{12}=g_{23}=g_{31}=0,$$

we have

$$\rho F^1=\rho F_r,\ \rho F^2=\frac{1}{r}\rho F_\varphi,\ \rho F^3=\rho F_3,\ \sigma^{11}=\sigma_{rr},\ \sigma^{12}=\frac{1}{r}\sigma_{r\varphi},$$

$$\sigma^{13}=\sigma_{r3},\ \sigma^{23}=\frac{1}{r}\sigma_{\varphi 3},\ \sigma^{22}=\frac{1}{r^2}\sigma_{\varphi\varphi},\ \sigma^{33}=\sigma_{33}. \qquad (2.28)$$

By (1.84),

$$\sigma^{mk}_{,\,m}=\frac{\partial\sigma^{mk}}{\partial x^m}+\Gamma^m_{jm}\sigma^{jk}+\Gamma^k_{jm}\sigma^{mj}, \qquad (2.29)$$

from which, when $k=1$, and from (2.28) we have

$$\sigma^{m1}_{,\,m}=\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{\partial\sigma_{3r}}{\partial x_3}+(\Gamma^1_{j1}+\Gamma^2_{j2}+\Gamma^3_{j3})\,\sigma^{j1}+$$
$$+\Gamma^1_{j1}\sigma^{1j}+\Gamma^1_{j2}\sigma^{2j}+\Gamma^3_{j3}\sigma^{3j}.$$

By using (1.64), we find

$$\Gamma^1_{11}=\Gamma^3_{13}=\Gamma^1_{21}=\Gamma^1_{31}=\Gamma^3_{23}=\Gamma^1_{23}=\Gamma^3_{33}=0,$$
$$\Gamma^2_{12}=\frac{1}{r},\qquad \Gamma^1_{22}=-r.$$

Then

$$\sigma^{m1}_{,\,m}=\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{\partial\sigma_{3r}}{\partial x_3}+\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}$$

Substituting this equality in (2.19), we obtain the first equation of motion

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{\partial\sigma_{3r}}{\partial x_3}+\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}+\rho F_r=\rho W_r. \qquad (2.30)$$

The other two equations of motion are derived in a similar way

$$\frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{\partial\sigma_{\varphi 3}}{\partial x_3}+\frac{2\sigma_{r\varphi}}{r}+\rho F_\varphi=\rho W_\varphi,$$

$$\frac{\partial\sigma_{r3}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\varphi 3}}{\partial\varphi}+\frac{\partial\sigma_{33}}{\partial x_3}+\frac{\sigma_{r3}}{r}+\rho F_3=\rho W_3.$$

The physical projections of the volume force and the acceleration in a spherical co-ordinate system are denoted, respectively, by ρF_r, ρF_φ, ρF_ψ and W_r, W_φ, W_ψ, and the physical projections of the stress tensor in the same coordinate system are denoted by σ_{rr}, $\sigma_{\varphi\varphi}$, $\sigma_{\psi\psi}$, $\sigma_{r\varphi}$, $\sigma_{\varphi\psi}$, $\sigma_{\psi r}$ (Fig. 9). In this co-ordinate system $g_{11} = H_1^2 = 1$,

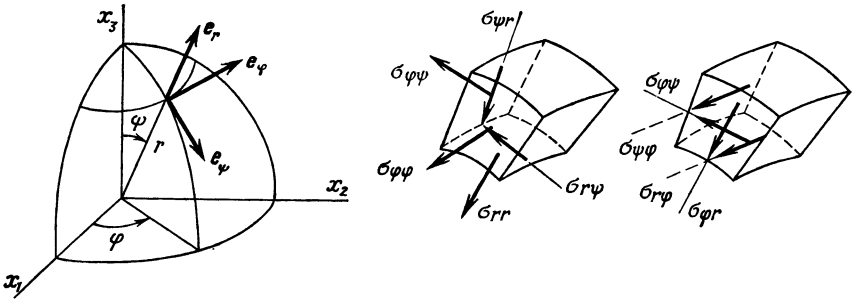

Fig. 9

$g_{33} = H_2^2 = r^2$, $g_{22} = H_3^2 = r^2 \sin^2 \psi$, $g_{12} = g_{23} = g_{31} = 0$. Taking into account these relations, from formulas (1.49) and (1.50) we find

$$\rho F^1 = \rho F_r, \quad \rho F^2 = \frac{1}{r \sin \psi} \rho F_\varphi, \quad \rho F^3 = \frac{1}{r} \rho F_\psi,$$

$$W^1 = W_r, \quad W^2 = \frac{W_\varphi}{r \sin \psi}, \quad W^3 = \frac{1}{r} W_\psi,$$

$$\sigma^{11} = \sigma_{rr}, \quad \sigma^{22} = \frac{1}{r^2 \sin^2 \psi} \sigma_{\varphi\varphi}, \quad \sigma^{33} = \frac{1}{r^2} \sigma_{\psi\psi},$$

$$\sigma^{12} = \frac{1}{r \sin \psi} \sigma_{r\varphi}, \quad \sigma^{23} = \frac{1}{r^2 \sin \psi} \sigma_{\varphi\psi}, \quad \sigma^{31} = \frac{1}{r} \sigma_{\psi r}.$$

From (2.19), (1.64), and (1.84) we obtain the equations of motion

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r \sin \psi} \frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{1}{r} \frac{\partial \sigma_{r\psi}}{\partial \psi} + \\ + \frac{1}{r} (2\sigma_{rr} + \sigma_{r\psi} \cot \psi - \sigma_{\varphi\varphi} - \sigma_{\psi\psi}) + \rho F_r = \rho W_r,$$

$$\frac{\partial \sigma_{r\psi}}{\partial r} + \frac{1}{r \sin \psi} \frac{\partial \sigma_{\varphi\psi}}{\partial \varphi} + \frac{1}{r} \frac{\partial \sigma_{\psi\psi}}{\partial \psi} + \\ + \frac{1}{r} [3\sigma_{r\psi} + (\sigma_{\psi\psi} - \sigma_{\varphi\varphi}) \cot \psi] + \rho F_\psi = \rho W_\psi, \quad (2.31)$$

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r \sin \psi} \frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{1}{r} \frac{\partial \sigma_{\varphi\psi}}{\partial \psi} + \\ + \frac{1}{r} (3\sigma_{r\varphi} + 2\sigma_{\varphi\psi} \cot \psi) + \rho F_\varphi = \rho W_\varphi.$$

Setting $W_r = W_\varphi = W_3 = 0$ and $W_r = W_\varphi = W_\psi = 0$ in Eqs. (2.30) and (2.31), we obtain the equations of equilibrium in terms of the components of the stress tensor, respectively, in a cylindrical and a spherical co-ordinate system.

14. DETERMINATION OF THE PRINCIPAL NORMAL STRESSES

Take a rectangular Cartesian co-ordinate system ox_k. The direction defined by the unit vector $\boldsymbol{n}$ with components $n_k = \cos(\boldsymbol{n}, x_k)$ is called the principal direction of the symmetric stress tensor σ_{rk} if the vector $\sigma_{rk}n_k$ is parallel to the vector $\boldsymbol{n}$, i.e.,

$$\sigma_{rk}n_k = \sigma n_r,$$

where σ is a scalar.

The last relation is written as

$$(\sigma_{rk} - \sigma\delta_{rk})\, n_k = 0. \tag{2.32}$$

Equalities (2.32) in n_k represent a linear homogeneous system of three equations. The condition for the existence of non-zero solutions is that the determinant of the coefficients of this system should be zero:

$$|\, \sigma_{kr} - \sigma\delta_{kr} \,| = 0. \tag{2.33}$$

Let us now prove that all of its roots are real; denote them by σ_l $(l = 1, 2, 3)$. Suppose the contrary: let $\sigma_l = \alpha_l + i\beta_l$ and the corresponding values $n_k = p_k + iq_k$; substitute this in (2.32). On comparing the real and imaginary parts, we obtain

$$(\sigma_{rk} - \alpha_l\delta_{rk})\, p_k + \beta_l\delta_{rk}q_k = 0,$$
$$(\sigma_{rk} - \alpha_l\delta_{rk})\, q_k - \beta_l\delta_{rk}p_k = 0.$$

Multiply the first equality by q_r and the second by p_r, and sum r from 1 to 3. By subtracting one result from the other, and taking into account the symmetry of σ_{rk}, δ_{rk}, we obtain

$$\beta_l\,(\delta_{rk}p_rp_k + \delta_{rk}q_rq_k) = 0. \tag{2.34}$$

Noting that p_k, q_k are not all zero and each term within the parentheses in (2.34) is positive, we come to the conclusion that $\beta_l = 0$. Consequently, the roots of Eq. (2.33) are always real and the corresponding values n_k^l, being the solutions of the system of linear equations with real coefficients (2.32), are also real. The quantities σ_l are called the principal components of the stress tensor, and n_k^l are their direction cosines.

Suppose that σ_l, σ_p are two distinct roots, and n_r^l, n_r^p are the corresponding values of n_r; from (2.32) we then find

$$(\sigma_{kr} - \sigma_l\delta_{kr})\, n_r^l = 0, \tag{2.35}$$
$$(\sigma_{kr} - \sigma_p\delta_{kr})\, n_r^p = 0. \tag{2.36}$$

Here the indices l and p are not summed. If (2.35) is multiplied by $\overset{p}{n}_k$ and (2.36) by $\overset{l}{n}_k$, then, by subtracting one result from the other, and taking into account the symmetry of σ_{kr} and δ_{kr}, we obtain

$$(\sigma_l - \sigma_p)\,\delta_{kr}\overset{l}{n}_r\overset{p}{n}_k = 0.$$

Since $\sigma_l \neq \sigma_p$, it follows that

$$\overset{l}{n}_k\overset{p}{n}_k = 0.$$

Thus, in this case the principal directions of the stress tensor are orthogonal and are uniquely determined. If Eq. (2.33) has two equal roots, say $\sigma_1 = \sigma_2$, the direction $\boldsymbol{n}^3$ corresponding to the third principal direction is perpendicular to the plane $\boldsymbol{n}^1$, $\boldsymbol{n}^2$.

Consequently, any two mutually orthogonal directions lying in a plane perpendicular to $\boldsymbol{n}^3$ may be taken as the corresponding principal directions. If, finally, all three principal stresses are equal, then any orthogonal directions may be taken as the principal directions.

The cubic equation (2.33) is now written as

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0. \tag{2.37}$$

According to the property of the roots of a cubic equation, the relations between the roots and the coefficients are as follows:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3,$$

$$I_2 = \begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_3 \end{vmatrix} + \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{vmatrix} + \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix} = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3,$$

$$I_3 = \begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix} = \sigma_1\sigma_2\sigma_3.$$

Take a matrix $\bar{C}$ in the form

$$\bar{C} = \|\bar{c}^i_j\| = \|\bar{\sigma}\delta^i_j - \bar{\sigma}^i_j\|.$$

The determinant of this matrix represents the left-hand side o Eq. (2.33) written in an arbitrary curvilinear co-ordinate system $\bar{x}^k$.

Consider continuous one-to-one transformations of co-ordinates $\bar{x}^k = \bar{x}^k(x^1, x^2, x^3)$. According to the transformation of mixed tensors (1.12) we have

$$c^i_j = \bar{c}^p_q \frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial x^i}{\partial \bar{x}^p}.$$

Introduce the notation

$$\frac{\partial \bar{x}^q}{\partial x^j} = a^q_j, \qquad \frac{\partial x^i}{\partial \bar{x}^p} = b^i_p;$$

then

$$C = \| c^i_j \| = \| \bar{c}^p_q b^i_p a^q_j \| = \| \bar{c}^p_q \| \, \| b^i_p \| \, \| a^q_j \| = B\bar{C}B^{-1}.$$

From this we conclude that the determinants of the matrices C and $\bar{C}$ are equal. Thus, the equation

$$|\sigma \delta^i_j - \sigma^i_j| = 0 \tag{2.38}$$

is invariant with respect to the choice of co-ordinate system and its roots always determine the principal components of the stress tensor. Consequently, the coefficients of Eq. (2.37) are invariants under a transformation of co-ordinates since they are completely determined by the roots, i.e., by the principal values of the stress tensor. By expanding (2.38), we obtain formulas for the invariants

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma^\alpha_\alpha,$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \frac{1}{2}\left[(\sigma^\alpha_\alpha)^2 - \sigma^\alpha_\beta \sigma^\beta_\alpha\right],$$

$$I_3 = \sigma_1\sigma_2\sigma_3 = |\sigma^\beta_\alpha|.$$

If the co-ordinate axes are taken coincident with the principal directions of the stress tensor, the components σ_{kr} ($k \neq r$) vanish in this co-ordinate system; the only non-zero stresses are the normal stresses σ_k acting on these planes.

CHAPTER III

Theory of strain

15. THE FINITE STRAIN TENSOR

Consider a continuous medium S in which a curvilinear co-ordinate system x^r ($r = 1, 2, 3$) is chosen. If a set of some functions of position determines the extension of any infinitesimal straight material segment passing through a given point, it is said that these functions determine the deformation of the neighbourhood of this point.

Let the co-ordinate lines chosen in the medium in its initial configuration be composed of material particles of the same medium. Suppose that during the deformation the co-ordinate lines continue to be made up of the same material particles. As a result of the deformation, the given co-ordinate system $Px^1x^2x^3$ with covariant base vectors e_k, being continually distorted together with the medium, assumes a certain position in one of the subsequent configurations $\bar{S}$. The configuration $\bar{S}$ may be taken as a new co-ordinate system $\bar{P}\bar{x}^1\bar{x}^2\bar{x}^3$ with base vectors $\bar{e}_k$. The reference system in which the displacement is determined is taken to be a co-ordinate system $ox_0^1x_0^2x_0^3$ with base vectors e_k^0 (Fig. 10). The system $ox_0^1x_0^2x_0^3$ may be chosen at will, and of the co-ordinate systems x^r and $\bar{x}^r$ only one may be chosen arbitrarily, i.e., if the system x^r is chosen, the system $\bar{x}^r$ is determined by the deformation, and vice versa.

According to formula (1.27), for the squares of line elements of the configurations S and $\bar{S}$ we have, respectively,

$$ds^2 = g_{nk}\,dx^n\,dx^k, \qquad d\bar{s}^2 = \hat{g}_{nk}\,d\bar{x}^n\,d\bar{x}^k. \tag{3.1}$$

Here g_{nk}, $\hat{g}_{nk}$ are the covariant metric tensors in S and $\bar{S}$, respectively; dx^n are the components of the infinitesimal vector PQ defining the position of the point Q relative to the point P, and $d\bar{x}^n$ are the components of the vector $\bar{P}\bar{Q}$ (see Fig. 10) which, by virtue of continuity, is infinitesimal.

The state of strain in a body is determined by the difference

$$d\bar{s}^2 - ds^2 = \hat{g}_{nk}\,d\bar{x}^n\,d\bar{x}^k - g_{nk}\,dx^n\,dx^k.$$

On the basis of (1.6) this difference may be put into the form

$$\overline{ds}^2 - ds^2 = 2\varepsilon_{nk}\, dx^n\, dx^k, \tag{3.2}$$

where

$$\varepsilon_{nk} = \frac{1}{2}(\bar{g}_{nk} - g_{nk}) \tag{3.3}$$

with

$$\bar{g}_{nk} = \hat{g}_{im}\frac{\partial \bar{x}^i}{\partial x^n}\,\frac{\partial \bar{x}^m}{\partial x^k}. \tag{3.4}$$

It is seen from (3.2) that ε_{nk} are the components of a symmetric covariant tensor of rank two, which is called the strain tensor. If all ε_{nk} are zero for all points, then $\overline{ds} = ds$ and the body undergoes no deformation. The extension of a line element ds_n along the co-ordinate line x^n is, by definition,

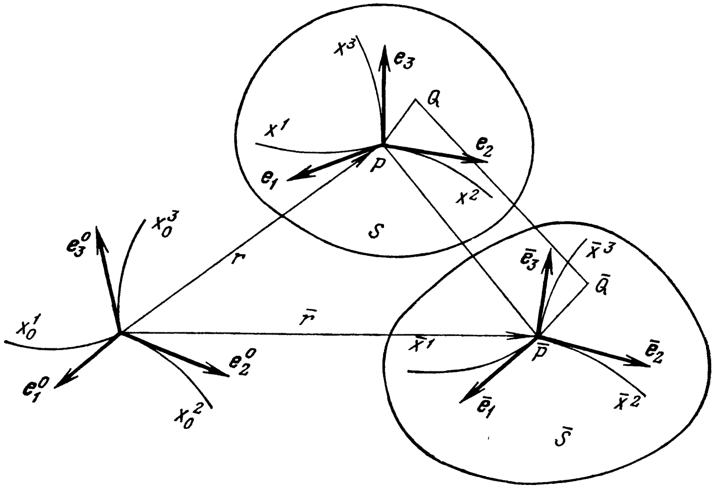

Fig. 10

$$e_n = \frac{\overline{ds}_n - ds_n}{ds_n}. \tag{3.5}$$

According to formulas (1.40)

$$\overline{ds}_n = \sqrt{\bar{g}_{nn}}\, dx^n, \quad ds_n = \sqrt{g_{nn}}\, dx^n. \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain

$$e_n = \frac{\overline{ds}_n - ds_n}{ds_n} = \sqrt{\frac{\bar{g}_{nn}}{g_{nn}}} - 1.$$

From this, using (3.3), we find

$$e_n = \sqrt{1 + \frac{2\varepsilon_{nn}}{g_{nn}}} - 1. \tag{3.7}$$

Here the index n is not summed; the root is taken with the plus sign since the extension is zero when $\varepsilon_{nn} = 0$.

The cosine of the angle $\bar{\theta}_{nk}$ between two line elements $d\bar{s}_n$ and $d\bar{s}_k$, which were directed along the co-ordinate lines x^n and x^k before deformation, is determined, according to formulas (1.54), (1.6), and (3.6), by the formula

$$\cos\bar{\theta}_{nk} = \frac{\hat{g}_{nk}\, d\bar{x}^n\, d\bar{x}^k}{d\bar{s}_n\, d\bar{s}_k} = \frac{\hat{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^n} \frac{\partial \bar{x}^j}{\partial x^k} dx^n\, dx^k}{\sqrt{\bar{g}_{nn}\bar{g}_{kk}}\, dx^n\, dx^k}.$$

By (3.4), this formula may be put into the form

$$\cos\bar{\theta}_{nk} = \frac{\bar{g}_{nk}}{\sqrt{\bar{g}_{nn}\bar{g}_{kk}}}.$$

After determining $\bar{g}_{nk}$ from (3.3) and substituting in the last formula, we find

$$\cos\bar{\theta}_{nk} = \frac{g_{nk} + 2\varepsilon_{nk}}{\sqrt{(g_{nn} + 2\varepsilon_{nn})(g_{kk} + 2\varepsilon_{kk})}}. \tag{3.8}$$

Formulas (3.7) and (3.8) show that the six components of the strain tensor defined by formulas (3.3) enable one to calculate completely the extensions along the co-ordinate lines, issuing from some point of a body, and the angle between two line elements after deformation, which were directed along the co-ordinate lines x^k before deformation. Since the angle between the co-ordinate lines before deformation is known, the change in this angle can thus be determined.

Let us now express the components of the strain tensor ε_{nk} in terms of the components of the displacement vector $\boldsymbol{u}$. It is seen from Fig. 10 that

$$\bar{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{u}. \tag{3.9}$$

Let $\boldsymbol{e}_n$, $\bar{\boldsymbol{e}}_n$ be the respective base vectors; hence, by (1.26),

$$\boldsymbol{e}_k = \frac{\partial \boldsymbol{r}}{\partial x^k}, \quad \bar{\boldsymbol{e}}_k = \frac{\partial \bar{\boldsymbol{r}}}{\partial \bar{x}^k}. \tag{3.10}$$

It follows from the vector equation (3.9) that

$$\frac{\partial \bar{\boldsymbol{r}}}{\partial x^k} = \frac{\partial \boldsymbol{r}}{\partial x^k} + \frac{\partial \boldsymbol{u}}{\partial x^k}.$$

Substituting (3.10) in the last relations, we have

$$\bar{e}_m \frac{\partial \bar{x}^m}{\partial x^k} = e_k + \frac{\partial u}{\partial x^k}. \qquad (3.11)$$

Since for the configurations S and $\bar{S}$, respectively,

$$g_{nk} = e_n \cdot e_k, \quad \hat{g}_{nk} = e_n \cdot e_k,$$

on the basis of the rules for scalar multiplication and formulas (3.4), (3.11) we find

$$\bar{g}_{nk} = \left(e_n + \frac{\partial u}{\partial x^n}\right)\left(e_k + \frac{\partial u}{\partial x^k}\right) =$$

$$= g_{nk} + e_n \cdot \frac{\partial u}{\partial x^k} + e_k \cdot \frac{\partial u}{\partial x^n} + \frac{\partial u}{\partial x^n} \frac{\partial u}{\partial x^k}. \qquad (3.12)$$

Substituting (3.12) in (3.3), we obtain

$$\varepsilon_{nk} = \frac{1}{2}\left(e_n \cdot \frac{\partial u}{\partial x^k} + e_k \cdot \frac{\partial u}{\partial x^n} + \frac{\partial u}{\partial x^n} \cdot \frac{\partial u}{\partial x^k}\right). \qquad (3.13)$$

Referred to the x^α co-ordinate system, the vector u is represented as

$$u = u^\alpha e_\alpha, \quad u = u_\alpha e^\alpha. \qquad (3.14)$$

Here the index α is summed. Taking into account (1.57), (1.79), (1.65), and (1.80), we find

$$\frac{\partial u}{\partial x^n} = u^\alpha_{,n} e_\alpha, \quad \frac{\partial u}{\partial x^k} = u_{\alpha,k} e^\alpha, \qquad (3.15)$$

where $u^\alpha_{,n}$, $u_{\alpha,k}$ are the covariant derivatives, respectively, of the contravariant and covariant vectors, equal to

$$u^\alpha_{,n} = \frac{\partial u^\alpha}{\partial x^n} + \Gamma^\alpha_{nm} u^m, \quad u_{\alpha,k} = \frac{\partial u_\alpha}{\partial x^k} - \Gamma^m_{\alpha k} u_m. \qquad (3.16)$$

It should be noted that here the Christoffel symbols must be calculated from the metric tensors for the configuration S.

Substituting (3.15) in (3.13), we have, finally,

$$\varepsilon_{nk} = \frac{1}{2}\left(u_{n,k} + u_{k,n} + u_{\alpha,k} u^\alpha_{,n}\right). \qquad (3.17)$$

By formulas (3.17), the components of the strain tensor are calculated through the covariant derivatives of the covariant and contravariant components of the displacement vector u in the system of the directions of the base vectors e^α and e_α.

16. THE SMALL STRAIN TENSOR

If the strains (extensions and shears) and the angles of rotation are small compared with unity and have the same order of smallness (as is the case in considering the deformation of bodies whose all dimensions are comparable in magnitude), the non-linear terms in the general formula (3.17) can be rejected as small quantities. In this case the strain tensor is called the small strain tensor and denoted by e_{nk}. Consequently,

$$e_{nk}=\frac{1}{2}(u_{n,k}+u_{k,n}). \tag{3.18}$$

The materials used in engineering, with the exception of rubber, some plastics and others, retain elasticity only at very small extensions and shears. This points clearly to the practical importance of the small strain tensor.

Let Q be a point close to the point P, and $\overline{Q}$ its position after displacement. In Fig. 10 $P\overline{P} = \boldsymbol{u}$ and $Q\overline{Q} = \boldsymbol{u}_1$ represent the displacements of the points P and Q, respectively.

By expanding $\boldsymbol{u}_1$ at the point P, and neglecting small quantities of higher order in dx^n (dx^n are the contravariant components of the vector PQ), we have

$$\boldsymbol{u}_1(Q)=\boldsymbol{u}(P)+\frac{\partial \boldsymbol{u}(P)}{\partial x^n}dx^n=\boldsymbol{u}(P)+\frac{\partial}{\partial x^n}(u^k\boldsymbol{e}_k)\,dx^n.$$

From the last relation we find the displacement $d\boldsymbol{u}$ of the point Q relative to the point P:

$$d\boldsymbol{u}=u^k_{\ ,n}\boldsymbol{e}_k\,dx^n$$

Substituting $d\boldsymbol{u} = du_k\boldsymbol{e}^k$, and multiplying scalarly both sides of the above equality by $\boldsymbol{e}_m$, we obtain

$$du_k = u_{k,n}dx^n, \tag{3.19}$$

where $u_{k,n}$ is the covariant derivative at the point P

Introducing the notation

$$\begin{aligned} e_{kn}&=\frac{1}{2}(u_{k,n}+u_{n,k}),\\ \omega_{kn}&=\frac{1}{2}(u_{k,n}-u_{n,k}), \end{aligned} \tag{3.20}$$

instead of (3.19) we have

$$du_k = (e_{kn} + \omega_{kn})\,dx^n$$

We conclude from formula (3.20) that ω_{kn} is an antisymmetric tensor called the rotation tensor. The displacement of the type $e_{kn}dx^n$

results from the deformation of the neighbourhood of the point *P*, while the displacement of the type $\omega_{kn} dx^n$ results from the rotation of the neighbourhood of the point *P* as an absolutely rigid body about this point.

17. STRAIN COMPATIBILITY EQUATIONS

As is clear from (3.17) or (3.18), the components of the strain tensor are not independent, they must satisfy some conditions. These conditions can be obtained on the assumption that the body in the undeformed configuration is in Euclidean space and continues to remain in it during deformation. As is known, the necessary and sufficient condition for this is that the Riemann-Christoffel tensor should be equal to zero for both the undeformed state S and the deformed state $\overline{S}$, i.e.,

$$R_{mnpq} = 0, \qquad \overline{R}_{mnpq} = 0$$

or

$$\overline{R}_{mnpq} - R_{mnpq} = 0. \tag{3.21}$$

Substituting the expressions for the components of the Riemann-Christoffel tensor (1.87) in (3.21), we obtain

$$\frac{1}{2}\left(\frac{\partial^2 \overline{g}_{mq}}{\partial x^n\, \partial x^p} + \frac{\partial^2 \overline{g}_{np}}{\partial x^m\, \partial x^q} - \frac{\partial^2 \overline{g}_{mp}}{\partial x^n\, \partial x^q} - \frac{\partial^2 \overline{g}_{nq}}{\partial x^m\, \partial x^p}\right) + \overline{g}^{ij}\left(\overline{\Gamma}_{i,\,np}\overline{\Gamma}_{j,\,mq} - \right.$$
$$\left. - \overline{\Gamma}_{i,\,nq}\overline{\Gamma}_{j,\,mp}\right) - \frac{1}{2}\left(\frac{\partial^2 g_{mq}}{\partial x^n\, \partial x^p} + \frac{\partial^2 g_{np}}{\partial x^m\, \partial x^q} - \frac{\partial^2 g_{mp}}{\partial x^n\, \partial x^q} - \frac{\partial^2 g_{nq}}{\partial x^m\, \partial x^p}\right) -$$
$$- g^{ij}\left(\Gamma_{i,\,np}\Gamma_{j,\,mq} - \Gamma_{i,\,nq}\Gamma_{j,\,mp}\right) = 0. \tag{3.22}$$

By using (3.3), and taking into account the fact that the Riemann-Christoffel tensor has only six independent components, R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{2123}, R_{3132}, from (3.22) we obtain six independent equations

$$\frac{\partial^2 \varepsilon_{mq}}{\partial x^n\, \partial x^p} + \frac{\partial^2 \varepsilon_{np}}{\partial x^m\, \partial x^q} - \frac{\partial^2 \varepsilon_{nq}}{\partial x^m\, \partial x^p} - \frac{\partial^2 \varepsilon_{mp}}{\partial x^n\, \partial x^q} - 2\varepsilon_{rs}\left(\Gamma^r_{qm}\Gamma^s_{pn} - \Gamma^r_{mp}\Gamma^s_{qn}\right) +$$
$$+ 2\Gamma^r_{np}\varepsilon_{mqr} + 2\Gamma^r_{qm}\varepsilon_{npr} - 2\Gamma^r_{nq}\varepsilon_{mpr} - 2\Gamma^r_{mp}\varepsilon_{nqr} = 0, \tag{3.23}$$

where

$$\varepsilon_{npr} = \frac{1}{2}\left(\frac{\partial \varepsilon_{pr}}{\partial x^n} + \frac{\partial \varepsilon_{rn}}{\partial x^p} - \frac{\partial \varepsilon_{np}}{\partial x^r}\right),$$

mnpq: 1212, 1313, 2323, 1213, 2123, 3132.

These equations indicate that the components of the strain tensor ε_{nk} are dependent. The equations that must be satisfied by the components of the strain tensor ε_{nk} are the necessary and sufficient conditions for the configurations S and $\overline{S}$ to belong to the Euclidean space.

18. THE STRAIN TENSOR REFERRED TO A CARTESIAN CO-ORDINATE SYSTEM

In a Cartesian co-ordinate system the covariant and contravariant vectors are identical; the covariant derivatives are also identical with the ordinary derivatives since in this case the metric tensor is constant and the Christoffel symbols are therefore equal to zero.

Thus, according to (3.17), the components of the finite strain tensor in a Cartesian co-ordinate system x_k are determined by the formulas

$$\varepsilon_{nk}=\frac{1}{2}\left(\frac{\partial u_n}{\partial x_k}+\frac{\partial u_k}{\partial x_n}+\frac{\partial u_\alpha}{\partial x_k}\frac{\partial u_\alpha}{\partial x_n}\right). \tag{3.24}$$

Remembering that in the case of an orthogonal Cartesian co-ordinate system $g_{nn} = 1$ and $g_{nk} = 0$, formulas (3.7) and (3.8) become

$$e_n=\sqrt{1+2\varepsilon_{nn}}-1, \quad \cos\overline{\theta}_{nk}=\frac{2\varepsilon_{kn}}{\sqrt{(1+2\varepsilon_{nn})(1+2\varepsilon_{kk})}}. \tag{3.25}$$

These formulas are used to calculate the extensions of line elements issuing from some point of a medium parallel to the axes of a rectangular Cartesian co-ordinate system, and the angles between these line elements after deformation.

According to (3.17) or (3.24), the small strain components in a rectangular Cartesian co-ordinate system x_k are

$$e_{kn}=\frac{1}{2}\left(\frac{\partial u_n}{\partial x_k}+\frac{\partial u_k}{\partial x_n}\right). \tag{3.26}$$

If small strains are considered, from (3.25) we have

$$e_n=1+\frac{1}{2}(2\varepsilon_{nn})+\ldots-1\cong e_{nn},$$

$$\cos\overline{\theta}_{kn}=2\varepsilon_{kn}\left\{1-\frac{1}{2}2(\varepsilon_{nn}+\varepsilon_{kk}+2\varepsilon_{nn}\varepsilon_{kk})+\ldots\right\}\cong 2e_{kn}.$$

Thus, it follows from $e_n = e_{nn}$ that e_{nn} are the extensions of line elements which were parallel to the corresponding axes of the rectangular Cartesian co-ordinate system before deformation. The quantities $2e_{kn}$ are the cosines of the angles formed after deformation between two line elements which were parallel to the co-ordinate axes before deformation.

We have

$$\cos\overline{\theta}_{kn} = \cos[90° - (\gamma_{kn} + \gamma_{nk})] = \sin(\gamma_{nk} + \gamma_{kn}) \cong \gamma_{kn} + \gamma_{nk}.$$

Here γ_{kn} is the angle of rotation towards the axis ox_n of a line element parallel to the axis ox_k and equal to $\frac{\partial u_n}{\partial x_k}$; γ_{nk} is the angle of rotation towards the axis ox_k of a line element parallel to the axis

ox_n and equal to $\frac{\partial u_k}{\partial x_n}$ (Fig. 11). Hence, $2e_{kn}$ is the change in the angle between two line elements parallel to the axes ox_n and ox_k $(k \neq n)$.

In a Cartesian co-ordinate system the components of the rotation tensor are

$$\omega_{kn}=\frac{1}{2}\left(\frac{\partial u_k}{\partial x_n}-\frac{\partial u_n}{\partial x_k}\right). \tag{3.27}$$

Rotate the axes ox_k of a rectangular Cartesian co-ordinate system; the new axes are denoted by ox'_k. Noting that e_{kn} is a tensor, on the basis of (1.13) we have $e'_{kn} = e_{mr}\alpha_{mk}\alpha_{rn}$, from which $e'_{kk} = e_{kk}$, i.e., the sum of the extensions in three mutually orthogonal directions issuing from the same point of a body is independent of their orientation at the given point.

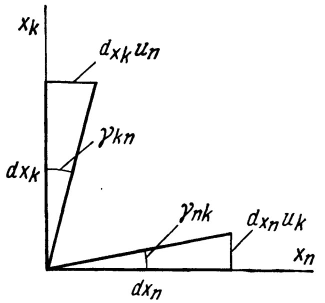

Fig. 11

19. COMPONENTS OF THE SMALL STRAIN AND ROTATION TENSORS REFERRED TO CYLINDRICAL AND SPHERICAL CO-ORDINATES

The physical projections of the displacement vector $\boldsymbol{u}$ in a cylindrical co-ordinate system (r, φ, x_3) are denoted by u_r, u_φ, u_3 and the physical components of the strain tensor by e_{rr}, $e_{\varphi\varphi}$, e_{33}, $e_{r\varphi}$, $e_{\varphi 3}$, e_{3r}. By using formulas (1.49) and (1.50), we find

$$\begin{array}{lll} u_1 = u_r, & u_2 = r u_\varphi, & u_3 = u_3, \\ e_{11} = e_{rr}, & e_{22} = r^2 e_{\varphi\varphi}, & e_{33} = e_{33}, \\ e_{12} = r e_{r\varphi}, & e_{23} = r e_{\varphi 3}, & e_{31} = e_{3r}. \end{array} \tag{3.28}$$

According to (3.20) and (3.28), for the six independent components of the small strain tensor and the rotation tensor we have

$$\begin{gathered} e_{rr}=\frac{\partial u_r}{\partial r}, \quad e_{\varphi\varphi}=\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_r}{r}, \quad e_{33}=\frac{\partial u_3}{\partial x_3}, \\ e_{r\varphi}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right), \\ e_{\varphi 3}=\frac{1}{2}\left(\frac{\partial u_\varphi}{\partial x_3}+\frac{1}{r}\frac{\partial u_3}{\partial \varphi}\right), \\ e_{3r}=\frac{1}{2}\left(\frac{\partial u_3}{\partial r}+\frac{\partial u_r}{\partial x_3}\right); \\ \omega_{r\varphi}=\omega_3=\frac{1}{2r}\left(\frac{\partial}{\partial r}(r u_\varphi)-\frac{\partial u_r}{\partial \varphi}\right), \end{gathered} \tag{3.29}$$

$$\omega_{\varphi 3}=\omega_r=\frac{1}{2r}\left(\frac{\partial u_3}{\partial \varphi}-r\frac{\partial u_\varphi}{\partial x_3}\right), \tag{3.30}$$
$$\omega_{3r}=\omega_\varphi=\frac{1}{2}\left(\frac{\partial u_r}{\partial x_3}-\frac{\partial u_3}{\partial r}\right).$$

In formulas (3.30) use has been made of the following abbreviated notation: $\omega_{r\varphi}=\omega_3$, $\omega_{\varphi 3}=\omega_r$, $\omega_{3r}=\omega_\varphi$.

Suppose that $u_3=0$, and u_φ and u_r are independent of the x_3 co-ordinate; from formulas (3.29) and (3.30) we find

$$e_{rr}=\frac{\partial u_r}{\partial r}, \quad e_{\varphi\varphi}=\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_r}{r},$$
$$e_{r\varphi}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right), \tag{3.31}$$
$$\omega_{r\varphi}=\omega_3=\frac{1}{2r}\left(\frac{\partial}{\partial r}(ru_\varphi)-\frac{\partial u_r}{\partial \varphi}\right).$$

These formulas define the three components of the strain tensor in the case of so-called plane strain with respect to the $r\varphi$ plane in polar co-ordinates.

The physical components of the displacement vector u in a spherical co-ordinate system (r, φ, ψ) are denoted by u_r, u_φ, u_ψ, and the physical components of the strain tensor in the same co-ordinate system by e_{rr}, $e_{\varphi\varphi}$, $e_{\psi\psi}$, $e_{r\varphi}$, $e_{\varphi\psi}$, $e_{\psi r}$. According to formulas (1.49) and (1.50) we have

$$\begin{aligned} &u_1=u_r, && u_2=ru_\varphi, && u_3=r\sin\psi u_\psi, \\ &e_{11}=e_{rr}, && e_{22}=r^2e_{\varphi\varphi}, && e_{33}=r^2\sin^2\psi e_{\psi\psi}, \\ &e_{12}=re_{r\varphi}, && e_{23}=r^2\sin\psi e_{\varphi\psi}, && e_{31}=r\sin\psi e_{\psi r}. \end{aligned} \tag{3.32}$$

Substituting (3.32) in (3.20), we find

$$e_{rr}=\frac{\partial u_r}{\partial r},$$
$$e_{\varphi\varphi}=\frac{1}{r\sin\psi}\frac{\partial u_\varphi}{\partial \varphi}+\frac{1}{r}u_r+\frac{1}{r\tan\psi}u_\psi,$$
$$e_{\psi\psi}=\frac{1}{r}\frac{\partial u_\psi}{\partial \psi}+\frac{1}{r}u_r, \tag{3.33}$$
$$e_{r\varphi}=\frac{1}{2}\left(\frac{1}{r\sin\psi}\frac{\partial u_r}{\partial \varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{1}{r}u_\varphi\right),$$
$$e_{\varphi\psi}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}-\frac{1}{r\tan\psi}u_\varphi+\frac{1}{r\sin\psi}\frac{\partial u_\psi}{\partial \varphi}\right),$$

$$\begin{aligned} e_{\psi r} &= \frac{1}{2}\left(\frac{\partial u_\psi}{\partial r} - \frac{1}{r}u_\psi + \frac{1}{r}\frac{\partial u_r}{\partial \psi}\right); \\ \omega_{\varphi\psi} = \omega_r &= \frac{1}{2r\sin\psi}\left(\frac{\partial u_\psi}{\partial\varphi} - \frac{\partial}{\partial\psi}(u_\varphi \sin\psi)\right), \\ \omega_{\psi r} = \omega_\varphi &= \frac{1}{2r}\left(\frac{\partial u_r}{\partial\psi} - \frac{\partial}{\partial r}(r u_\psi)\right), \\ \omega_{r\varphi} = \omega_\psi &= \frac{1}{2r\sin\psi}\left(\frac{\partial}{\partial r}(r u_\varphi \sin\psi) - \frac{\partial u_r}{\partial\varphi}\right). \end{aligned} \tag{3.34}$$

20. PRINCIPAL EXTENSIONS

By analogy with the theory of stress, the principal directions of the strain tensor in a rectangular Cartesian co-ordinate system are defined as the directions for which the following conditions are fulfilled:

$$\varepsilon_{kn}\alpha_n = \varepsilon\alpha_k. \tag{3.35}$$

Here ε is a scalar, α_k are the direction cosines of the unit vector $\mathbf{v}$.

By using the Kronecker symbols, the system of equations (3.35) is written as

$$(\varepsilon_{kn} - \delta_{kn}\varepsilon)\,\alpha_n = 0. \tag{3.36}$$

Since the cosines cannot all be zero simultaneously, we have

$$|\, \varepsilon_{kn} - \delta_{kn}\varepsilon \,| = 0. \tag{3.37}$$

The principal values of the strain tensor, which are called the principal extensions, are the roots of the cubic equation (3.37). The directions corresponding to the principal extensions ε_1, ε_2, ε_3 are mutually perpendicular. When two of the roots are equal, the directions corresponding to these roots lie in a plane perpendicular to the direction corresponding to the simple root; in this case any mutually orthogonal directions lying in this plane may be taken as principal. If all three roots are equal, then any perpendicular directions may be taken as principal.

The cubic equation (3.37) is written in the form

$$\varepsilon^3 - I_1\varepsilon^2 + I_2\varepsilon - I_3 = 0.$$

According to the property of the roots of a cubic equation, the relations between the roots and the coefficients are as follows:

$$\begin{aligned} I_1 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ I_2 &= \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_3\varepsilon_2, \\ I_3 &= \varepsilon_1\varepsilon_2\varepsilon_3. \end{aligned}$$

By analogy with the theory of stress, I_1, I_2, I_3 are invariants. Thus,

$$I_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_\alpha^\alpha,$$

$$I_2 = \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 = \frac{1}{2}\left[(\varepsilon_\alpha^\alpha)^2 - \varepsilon_\alpha^\beta \varepsilon_\beta^\alpha\right],$$

$$I_3 = \varepsilon_1\varepsilon_2\varepsilon_3 = |\varepsilon_\alpha^\beta|.$$

The first invariant of the strain tensor in the case of small strains represents the unit change of volume. Indeed, take the principal axes of the strain tensor at a point P of the medium. Construct on these axes a parallelepiped having edges equal to dx_k before deformation. After deformation the parallelepiped under consideration, remaining rectangular, will have edges $(1 + e_k)\,dx_k$. The volume of the parallelepiped before deformation is $d\tau = dx_1 dx_2 dx_3$, after deformation

$$d\tau_1 = (1 + e_1)(1 + e_2)(1 + e_3)\,dx_1 dx_2 dx_3 = (1 + e_1)(1 + e_2)(1 + e_3)\,d\tau.$$

Rejecting small quantities of higher order, we have

$$d\tau_1 = (1 + e_1 + e_2 + e_3)\,d\tau.$$

Denoting the unit change of volume at the point P, or the volume strain, by θ, we obtain

$$\theta = \frac{d\tau_1 - d\tau}{d\tau} = e_1 + e_2 + e_3.$$

Remembering that $I_1 = e_1 + e_2 + e_3 = e_{11} + e_{22} + e_{33}$, for the volume strain we obtain

$$\theta = I_1 = e_{11} + e_{22} + e_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \operatorname{div} \boldsymbol{u}, \tag{3.38}$$

i.e., $\theta = I_1$ is the volume strain at the point P.

If the co-ordinate axes are taken coincident with the principal directions of the strain tensor, the components e_{kr} $(k \neq r)$ vanish in this co-ordinate system, and only extensions e_k acting on these planes will remain.

21. STRAIN COMPATIBILITY EQUATIONS IN SOME CO-ORDINATE SYSTEMS (SAINT VENANT'S CONDITIONS)

In view of the fact that in the case of a Cartesian co-ordinate system the Christoffel symbols are identically zero, the compatibility equations (3.23) in this system assume the form

$$\frac{\partial^2 \varepsilon_{mq}}{\partial x_n\, \partial x_p} + \frac{\partial^2 \varepsilon_{np}}{\partial x_m\, \partial x_q} - \frac{\partial^2 \varepsilon_{nq}}{\partial x_m\, \partial x_p} - \frac{\partial^2 \varepsilon_{mp}}{\partial x_n\, \partial x_q} = 0, \tag{3.39}$$

where $mnpq$: 1212, 1313, 2323, 1213, 2123, 3132.

The compatibility equations in a Cartesian co-ordinate system were obtained by Saint Venant for small strains directly by eliminating the displacement components from formulas (3.26).

If Saint Venant's conditions are fulfilled for an arbitrary tensor e_{ij}, a displacement field can be found for which e_{ij} is the strain tensor. In the case of a simply connected body the displacement is determined to within a rigid-body displacement, in the case of a multiply connected body some additional conditions must be fulfilled.

To obtain the strain compatibility equations in a cylindrical co-ordinate system, we take account of formulas (3.28) and (1.64) in Eqs. (3.23); after some computations we obtain

$$\begin{gathered}
\frac{1}{r}\frac{\partial^2 e_{rr}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial e_{\varphi\varphi}}{\partial r}\right)-\frac{\partial e_{rr}}{\partial r}-\frac{2}{r}\frac{\partial^2(re_{\varphi r})}{\partial r\,\partial\varphi}=0,\\
\frac{\partial^2 e_{rr}}{\partial x_3^2}+\frac{\partial^2 e_{33}}{\partial r^2}-2\frac{\partial^2 e_{r3}}{\partial r\,\partial x_3}=0,\\
\frac{\partial^2 e_{\varphi\varphi}}{\partial x_3^2}+\frac{1}{r^2}\frac{\partial^2 e_{33}}{\partial\varphi^2}+\frac{1}{r}\frac{\partial e_{33}}{\partial r}-\frac{2}{r}\frac{\partial}{\partial x_3}\left(\frac{\partial e_{\varphi 3}}{\partial\varphi}+e_{r3}\right)=0,\\
-\frac{1}{r}\frac{\partial^2 e_{rr}}{\partial\varphi\,\partial x_3}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(re_{\varphi 3})}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2(r^2e_{r\varphi})}{\partial r\,\partial x_3}-\frac{\partial^2}{\partial r\,\partial\varphi}\left(\frac{e_{r3}}{r}\right)=0,\\
r\frac{\partial}{\partial x_3}\left(e_{rr}-\frac{\partial(re_{\varphi\varphi})}{\partial r}\right)-\frac{\partial^2 e_{r3}}{\partial\varphi^2}+\frac{\partial^2(re_{\varphi 3})}{\partial r\,\partial\varphi}+\frac{\partial^2(re_{\varphi r})}{\partial x_3\,\partial\varphi}=0,\\
\frac{\partial^2}{\partial\varphi\,\partial r}\left(\frac{e_{33}}{r}\right)+\frac{\partial^2 e_{r\varphi}}{\partial x_3^2}-r\frac{\partial^2}{\partial r\,\partial x_3}\left(\frac{e_{\varphi 3}}{r}\right)-\frac{1}{r}\frac{\partial^2 e_{r3}}{\partial\varphi\,\partial x_3}=0.
\end{gathered}\tag{3.40}$$

22. DETERMINATION OF DISPLACEMENTS FROM THE COMPONENTS OF THE SMALL STRAIN TENSOR

Formulas (3.26) are used to calculate the components of the small strain tensor when displacements u_k (x_1, x_2, x_3) are given in a rectangular Cartesian co-ordinate system. To calculate the latter when the components of the strain tensor e_{kn} are given, it is necessary to solve the system of six first-order linear partial differential equations (3.26). For the system to be consistent, the given components e_{kn} must satisfy the so-called compatibility conditions, or the conditions for the integrability of this system. Assume that e_{kn} are given single-valued functions of x_k having continuous second-order partial derivatives.

If displacements u_k are found from the given components of the strain tensor, by adding to them an arbitrary infinitesimal rigid-body displacement of the body as a whole, we obtain new displacements obviously also corresponding to the given components of the strain tensor since the rigid-body displacement has no effect on pure strain. Hence, for definiteness we may, for example, assign, in addition, the projections of the displacement vector of some point of the body and the components of the rotation tensor at this point.

The region of assignment of the components of the strain tensor, which are used to find the projections of the displacement vector in the same region (occupied by the body before deformation), is denoted by τ. Assume, for the present, that this region is simply connected. From (3.26) and (3.27) we have

$$\frac{\partial u_k}{\partial x_n} = e_{kn} + \omega_{kn}. \tag{3.41}$$

Let the components of displacement $\overset{\circ}{u}_n$ and the components of the rotation tensor $\overset{\circ}{\omega}_{nk}$ be given at a point M° $(\overset{\circ}{x}_1, \overset{\circ}{x}_2, \overset{\circ}{x}_3)$. The components of displacement at a point M' (x'_1, x'_2, x'_3) are, from (3.41),

$$u_k = \overset{\circ}{u}_k + \int\limits_{M^\circ}^{M'} (e_{kn} + \omega_{kn})\, dx_n. \tag{3.42}$$

Here the integration is carried out along an arbitrary curve joining the points M° and M' and lying entirely within the region τ, x_n are the current co-ordinates of a point of this curve; consequently,

$$dx_n = -d\,(x'_n - x_n)$$

Substituting the last relation in (3.42) gives

$$u_k = \overset{\circ}{u}_k + \int\limits_{M^\circ}^{M'} e_{kn}\, dx_n - \int\limits_{M^\circ}^{M'} \omega_{kn} d\,(x'_n - x_n).$$

By integrating the last integral by parts, we obtain

$$u_k = \overset{\circ}{u}_k + \int\limits_{M^\circ}^{M'} e_{kn}\, dx_n - \omega_{kn}(x'_n - x_n)\Big|_{M^\circ}^{M'} + \int\limits_{M^\circ}^{M'} (x'_n - x_n)\frac{\partial \omega_{kn}}{\partial x_m}\, dx_m. \tag{3.43}$$

Here the index m is also summed. By (3.27),

$$\frac{\partial \omega_{kn}}{\partial x_m} = \frac{1}{2}\frac{\partial}{\partial x_m}\left(\frac{\partial u_k}{\partial x_n} - \frac{\partial u_n}{\partial x_k}\right) = \frac{\partial}{\partial x_n}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right) - \\ - \frac{\partial}{\partial x_k}\frac{1}{2}\left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n}\right).$$

By using (3.26), we find

$$\frac{\partial \omega_{kn}}{\partial x_m} = \frac{\partial}{\partial x_n} e_{km} - \frac{\partial}{\partial x_k} e_{nm}.$$

Substituting this in the last integral of (3.43), we obtain, finally,

$$u_k = \overset{\circ}{u}_k + \overset{\circ}{\omega}_{kn}(x'_n - \overset{\circ}{x}_n) + \int\limits_{M^\circ}^{M'} \left[e_{km} + (x'_n - x_n)\left(\frac{\partial e_{km}}{\partial x_n} - \frac{\partial e_{nm}}{\partial x_k}\right)\right] dx_m. \tag{3.44}$$

By their physical meaning, u_k must be independent of the path of integration $M^\circ M'$; for this, in the case of a simply connected region it is necessary and sufficient that the integrand be the total differential at all points (x_1, x_2, x_3) and for all values (x_1', x_2', x_3') in the region τ. These conditions lead to the following relations:

$$\frac{\partial^2 e_{mn}}{\partial x_k\,\partial x_r}+\frac{\partial^2 e_{kr}}{\partial x_m\,\partial x_n}-\frac{\partial^2 e_{kn}}{\partial x_m\,\partial x_r}-\frac{\partial^2 e_{mr}}{\partial x_k\,\partial x_n}=0, \qquad (3.45)$$

where *mkrn*: 1212, 1313, 2323, 1213, 2123, 3132.

Thus, relations (3.45) ensure the consistency of six differential equations (3.26) for the determination of three functions u_k. These equations are identical with Saint Venant's compatibility conditions; hence, Saint Venant's conditions also ensure the integrability of six differential equations (3.26). With Saint Venant's conditions, formulas (3.44) determine u_k whatever the shape of the curve of integration lying entirely within the region τ.

If the body is multiply connected, the integral in formula (3.44) can, in general, receive finite increments, in which case the uniqueness of displacements is not ensured, whereas they must be unique. By means of suitable imaginary cuts a multiply connected body can be transformed into a simply connected body; if Saint Venant's strain compatibility conditions are fulfilled, the displacements u_k determined by (3.44) will then be single-valued functions provided that the curve of integration nowhere crosses the lines of cuts. As the point M' approaches some point of the line of a cut from the left or right, u_k will, in general, assume different values. It appears from the above that in the case of a multiply connected region the additional conditions for the compatibility of strains are $(u_k)_{\text{left}} = (u_k)_{\text{right}}$ along all lines of cuts.

The most general tensor presentation of the theory of stress and strain for an arbitrary co-ordinate system is of particular value for finite deformations. The general equations and formulas derived above will enable us to develop them subsequently in appropriate co-ordinate system.

In the following discussion we shall mostly use a rectangular Cartesian co-ordinate system.

CHAPTER IV

Stress-strain relations

23. GENERALIZED HOOKE'S LAW

The equations obtained in Chaps. II and III are not sufficient to determine the states of stress and strain produced in a body by applied forces. These equations must therefore be supplemented by certain relations connecting the states of stress and strain. These relations are determined from the physical properties of a solid undergoing deformation. The establishment of the stress-strain relation is an important problem of continuum mechanics requiring the carrying-out of preliminary experiments. This relation is usually idealized by simple mathematical formulas.

During deformation, the removal of the external forces leads in some cases to complete recovery of a body to the natural state, i.e., the strain is recoverable, while in other cases the body, on removing the load, retains the strains, called permanent or plastic strains, i.e., the strain is irrecoverable. The following discussion will be concerned with fully recoverable small strains.

Assume that at each point of the body under consideration there is a one-to-one correspondence between the states of stress and strain.

By expressing this analytically, we obtain six relations of the form

$$\sigma^{ij}=f^{ij}(e_{kl}), \tag{4.1}$$

which are uniquely solvable for the components of the strain tensor

$$e^{ij}=\varphi^{ij}(\sigma_{kl}). \tag{4.2}$$

The undeformed state of an elastic body is taken to be a state in which there are no stresses. This state will be further used as the origin of stress and strain. Hence, the functions f^{ij} and also φ^{ij} vanish when their arguments become zero:

$$f^{ij}(0)=\varphi^{ij}(0)=0.$$

For many materials, relations (4.1) and (4.2) are linear if the magnitudes of the stresses are confined to a certain range.

The linear law for the relation between stress and strain is called the generalized Hooke's law. The general form of writing Hooke's law is as follows:

$$\sigma^{ij}=C^{ijkl}e_{kl}. \tag{4.3}$$

Interchange in turn the superscripts *ijkl* in (4.3) to *jikl*, *ijlk*, *jilk*; from the symmetry of the stress tensor and the strain tensor we reveal the symmetry of the quantities

$$C^{ijkl}=C^{jikl}=C^{ijlk}=C^{jilk}, \tag{4.4}$$

which form a tensor of rank four. The quantities C^{ijkl} are called the elastic coefficients of a body. The total number of different coefficients C^{ijkl}, as may be ascertained from (4.4), is 36. The elastic coefficients depend on the metric tensor g_{ij} of the undeformed body and on its physical properties. Noting that

$$g^{\alpha}_{m}=g_{km}g^{k\alpha}=\begin{cases}1 \text{ when } \alpha=m,\\ 0 \text{ when } \alpha\neq m,\end{cases}$$

(4.3) is written as

$$\sigma^{ij}=C^{ijkl}e_{kl}g_{km}g^{km}g_{ln}g^{ln}.$$

According to the rule for scalar multiplication, $C^{ijkl}g_{km}g_{ln}$ is a mixed tensor C^{ij}_{mn}, and $e_{kl}g^{km}g^{ln}$ are the contravariant components of the small strain tensor.

Thus, instead of (4.3) we have

$$\sigma^{ij}=C^{ij}_{mn}e^{mn}. \tag{4.5}$$

We write Hooke's law (4.3) as $\sigma_{ij}=C_{ijkl}e^{kl}$ $(C_{ijkl}=C_{jikl}=$ $=C_{ijlk}=C_{jilk}$ by virtue of the symmetry of σ_{ij} and $e^{kl})$ and make transformations similar to those applied above. We then obtain

$$\sigma_{ij}=C^{mn}_{ij}e_{mn}. \tag{4.6}$$

For the case of a prismatic rod loaded axially in tension Hooke's law is written as

$$e=C\sigma,\quad e'=C'\sigma,$$

where e and e' are, respectively, the longitudinal and lateral strains, C and C' are constants equal to

$$C=\frac{1}{E},\quad C'=-\frac{\nu}{E}.$$

Here E is the longitudinal modulus of elasticity, ν is Poisson's ratio.

24. WORK DONE BY EXTERNAL FORCES

Let an elastic body be acted on by surface forces $\boldsymbol{T}_n$ and volume forces $\rho\boldsymbol{F}$. Suppose these forces are given increments $d\boldsymbol{T}_n$ and $\rho d\boldsymbol{F}$, respectively. In consequence the displacement vector $\boldsymbol{u}$ changes to $d\boldsymbol{u}$. The work dV done by the forces $\boldsymbol{T}_n$ and $\rho\boldsymbol{F}$ during the additional deformation of the body is

$$dV = \int_{\tau} \rho F_k \, du_k \, d\tau + \int_{\omega} T_{nk} \, du_k \, d\omega,$$

where τ is the volume of the body and ω its surface.

Noting that $T_{nk} = \sigma_{kr} n_r$, and transforming the surface integral into a volume integral, for dV we obtain

$$dV = \int_{\tau} \rho F_k \, du_k \, d\tau + \int_{\tau} \frac{\partial}{\partial x_r} (\sigma_{kr} \, du_k) \, d\tau$$

or

$$dV = \int_{\tau} \left(\rho F_k + \frac{\partial \sigma_{kr}}{\partial x_r} \right) du_k \, d\tau + \int_{\tau} \sigma_{kr} \frac{\partial}{\partial x_r} (du_k) \, d\tau.$$

In view of the fact that

$$\frac{\partial \sigma_{kr}}{\partial x_r} + \rho F_k = 0, \qquad \frac{\partial}{\partial x_r} (du_k) = d \frac{\partial u_k}{\partial x_r}$$

we have

$$dV = \int_{\tau} \sigma_{kr} d \frac{\partial u_k}{\partial x_r} \, d\tau. \tag{4.7}$$

In the integrand of (4.7) the indices k and r are to be summed. Because of the symmetry of the stress tensor σ_{kr}, we have

$$\sigma_{kr} d \frac{\partial u_k}{\partial x_r} = \sigma_{kr} d \frac{1}{2} \left(\frac{\partial u_k}{\partial x_r} + \frac{\partial u_r}{\partial x_k} \right) = \sigma_{kr} \, de_{kr}.$$

On the basis of the last relation we finally obtain for dV:

$$dV = \int_{\tau} \sigma_{kr} \, de_{kr} \, d\tau. \tag{4.8}$$

In statics this formula determines the work done by the external forces ρF and T_n during the increments of the components of the strain tensor produced by the change in the above forces; the work per unit volume is

$$dA = \sigma_{kr} de_{kr}. \tag{4.9}$$

25. STRESS TENSOR POTENTIAL

According to W. Thomson's idea, the first and second laws of thermodynamics are applicable to the study of the deformation process in a body. Suppose that the deformation process in a body is thermodynamically reversible; the state of the body is then uniquely determined by the thermodynamic variables.

If U, Q, A, S are, respectively, the internal energy, the quantity of heat, the work done by external forces, and the entropy per unit volume of the body, in the case of small strains we have, by the first and second laws of thermodynamics,

$$dU = dQ + dA, \quad dQ = T\,dS. \tag{4.10}$$

Here dU and dS are, respectively, the increments of internal energy and entropy representing the total differentials of the independent thermodynamic variables defining the state of the body, T is the absolute temperature.

Eliminating the increment of heat dQ from (4.10), and using (4.9), we obtain the fundamental thermodynamic relation for the deformation process in a body:

$$dU = T\,dS + \sigma_{kr} de_{kr}. \tag{4.11}$$

We define an elastic body so that the specification of the strain tensor e_{kr} and of one thermodynamic variable (temperature T or entropy S) will completely determine its state, i.e., the stress tensor σ_{kr} and the thermodynamic potentials U and $F = U - TS$ (the latter is termed the Helmholtz free energy).

The independent variables determining the state of an elastic body are chosen as e_{kr} and the temperature T. The Helmholtz free energy is then a function of e_{kr} and T only, i.e.,

$$F = F(e_{kr}, T). \tag{4.12}$$

After determining U from

$$F = U - TS, \tag{4.13}$$

and substituting it in (4.11), we find

$$dF = \sigma_{kr} de_{kr} - S dT. \tag{4.14}$$

On the other hand, from (4.12) we have

$$dF = \frac{\partial F}{\partial e_{kr}} de_{kr} + \frac{\partial F}{\partial T} dT. \tag{4.15}$$

On comparing the coefficients of like differentials in (4.14) and (4.15), we obtain

$$\sigma_{kr} = \frac{\partial F}{\partial e_{kr}^*}, \quad S = -\frac{\partial F}{\partial T}. \tag{4.16}$$

Here

$$e_{kk}^* = e_{kk}, \quad e_{kr}^* = 2e_{kr} \qquad (k \neq r).$$

The first relation of (4.16) shows that for the given choice of the independent thermodynamic variable the Helmholtz free energy is the stress tensor potential for an elastic body. If the independent thermodynamic variable is chosen to be the entropy, i.e., if we suppose that the state of an elastic body is completely determined by specifying the strain tensor e_{kr} and the entropy S, then $U=U(e_{kr}, S)$ and therefore

$$dU=\frac{\partial U}{\partial e_{kr}}de_{kr}+\frac{\partial U}{\partial S}dS. \tag{4.17}$$

On comparing (4.11) and (4.17), we obtain

$$\sigma_{kr}=\frac{\partial U}{\partial e^*_{kr}}, \quad T=\frac{\partial U}{\partial S}. \tag{4.18}$$

The first relation of (4.18) proves that the internal energy is the stress tensor potential.

In the case of an adiabatic process, i.e., when $dQ = 0$, it follows from the first relation of (4.10) that dA is the total differential of the independent variables e_{kr}, i.e.,

$$dA=\frac{\partial A}{\partial e_{kr}}de_{kr}.$$

On the other hand, by (4.9), we have

$$dA = \sigma_{kr}de_{kr},$$

from which

$$\sigma_{kr}=\frac{\partial A}{\partial e^*_{kr}}. \tag{4.19}$$

If an isothermal process (T = constant) takes place, by virtue of the second relation of (4.10) the increment of heat dQ, as well as dU, is the total differential. Consequently, we conclude from the first relation of (4.10) that dA is the total differential. Then

$$\sigma_{kr}=\frac{\partial A}{\partial e^*_{kr}}. \tag{4.20}$$

As seen from (4.19) and (4.20), in the case of adiabatic and isothermal quasi-static processes the work done by external forces is the stress tensor potential and it can be determined from the equality

$$dA=\sigma_{kr}\,de_{kr}=\frac{\partial A}{\partial e_{kr}}de_{kr}. \tag{4.21}$$

If the body is linearly elastic, the quantities $\frac{\partial A}{\partial e_{kr}}$ are, according to (4.6), linear and homogeneous in the components of the strain tensor e_{kr}. Hence, A is a second-degree homogeneous polynomial in e_{kr}. Consequently, by Euler's theorem on homogeneous functions, we have

$$\frac{\partial A}{\partial e_{kr}}e_{kr}=2A.$$

This relation is known as Clapeyron's formula.

The independent variables determining the state of an elastic body are now taken to be the stress tensor σ_{kr} and the temperature T.

Introduce a new function

$$J = U - TS - \sigma_{kr} e_{kr}, \tag{4.22}$$

which depends only on σ_{kr} and T; then

$$dJ = \frac{\partial J}{\partial \sigma_{kr}} d\sigma_{kr} + \frac{\partial J}{\partial T} dT. \tag{4.23}$$

On the other hand, from (4.9), (4.11), and (4.22), we have

$$dJ = -S\, dT - e_{kr} d\sigma_{kr}. \tag{4.24}$$

Comparison of (4.23) and (4.24) leads to the expressions

$$e^*_{kr} = -\frac{\partial J}{\partial \sigma_{kr}}, \quad S = -\frac{\partial J}{\partial T}. \tag{4.25}$$

Thus, in the case when the independent variables are chosen as σ_{kr} and T, function (4.22) is the potential for the strain tensor of an elastic body. It can easily be shown that, in the independent variables σ_{kr} and T, the function

$$\sigma_{kr} e_{kr} - A \tag{4.26}$$

is the strain tensor potential for adiabatic and isothermal deformation processes in a body.

Formula (4.9) can be rearranged in the form

$$dA = \sigma_{kr} de_{kr} = d(\sigma_{kr} e_{kr}) - e_{kr} d\sigma_{kr}.$$

From this, remembering that for adiabatic and isothermal deformation processes dA is the total differential, we have

$$d(A - \sigma_{kr} e_{kr}) = -e_{kr}\, d\sigma_{kr}.$$

Consequently,

$$e^*_{kr} = \frac{\partial(\sigma_{kr} e_{kr} - A)}{\partial \sigma_{kr}}.$$

Thus, the function $\sigma_{kr} e_{kr} - A$ is the strain tensor potential for adiabatic and isothermal processes.

If the body is linearly elastic, by Clapeyron's formula $\sigma_{kr} e_{kr} = 2A$, and by (4.26) the strain tensor potential, called the elastic potential, is A. Consequently,

$$e^*_{kr} = \frac{\partial A}{\partial \sigma_{kr}}. \tag{4.27}$$

These relations are known as Castigliano's formulas and are valid for adiabatic and isothermal processes in linearly elastic bodies.

26. POTENTIAL IN THE CASE OF A LINEARLY ELASTIC BODY

Suppose that the deformation process in a solid takes place adiabatically or isothermally; then relation (4.19) holds. With this relation, dA is the total differential of a continuous single-valued function A depending only on the strain tensor e_{kr}. Assume the body to be linearly elastic. Substitute the expressions of the generalized Hooke's law (4.6) in (4.9):

$$dA = C_{ij}^{mn} e_{mn}\, de_{ij}.$$

For the right-hand side of this equality to be also the total differential, the condition $C_{mn}^{ij} = C_{ij}^{mn}$ must be fulfilled. Taking this into account, upon integration we find

$$A = \frac{1}{2} C_{ij}^{mn} e_{mn} e_{ij}. \qquad (4.28)$$

27. VARIOUS CASES OF ELASTIC SYMMETRY OF A BODY

The theory of elasticity deals with homogeneous and non-homogeneous, isotropic and anisotropic bodies. A homogeneous body is one whose elastic properties are the same at all of its points; an isotropic body is one whose elastic properties are the same in all directions. Otherwise the body is said to be non-homogeneous and anisotropic. An example of anisotropic bodies is provided by crystals.

Metals and their alloys used in engineering have polycrystalline structure in the form of randomly oriented crystal grains. A polycrystal whose size is of the same order of magnitude as the size of crystal grains is by its nature non-homogeneous and anisotropic. In comparing specimens whose dimensions considerably exceed the size of individual grains, in view of the arbitrariness of the orientation of grains and the smallness of their size in comparison with the specimen dimensions (from fractions of a micron to tens of microns), a polycrystal behaves as a homogeneous and isotropic continuous medium.

It should be noted that the manufacturing processes and various kinds of mechanical treatment introduce more or less significant anisotropy and inhomogeneity into a metal; hence, there is always only approximate homogeneity and isotropy of materials.

The symmetry of the structure of anisotropic bodies leads to relations among the elastic coefficients. We shall consider some special cases of elastic symmetry.

For anisotropic linearly elastic bodies when the deformation process takes place isothermally or adiabatically, in view of the fact that $C_{ij}^{mn} = C_{mn}^{ij}$ the number of elastic coefficients is 21.

Let the body have one plane of elastic symmetry, which is taken as the ox_1x_2 plane. If the direction of the ox_3 axis is reversed, the signs of x_3 and u_3 must be reversed, and hence the signs of the strain components e_{31}, e_{23} are also changed.

The reversal of the direction of the ox_3 axis must not change the magnitude of the elastic potential A since it is an invariant. In formula (4.28) the first degree terms in e_{23}, e_{13} must therefore vanish, i.e.,

$$C^{13}_{11}=C^{23}_{11}=C^{22}_{23}=C^{33}_{13}=C^{33}_{23}=C^{23}_{12}=C^{13}_{12}=C^{31}_{22}=0. \qquad (4.29)$$

Thus, in the case when the body has one plane of symmetry of elastic properties, the number of elastic constants reduces to 13.

Let the body have two mutually perpendicular planes of symmetry of elastic properties. These planes are taken as the co-ordinate planes ox_1x_2 and ox_2x_3. If the magnitude of the elastic potential is to remain unaltered when the direction of the ox_2 axis is reversed, in which case the sign of the component e_{12} is changed, in addition to conditions (4.29) we must set

$$C^{11}_{12}=C^{22}_{12}=C^{33}_{12}=C^{13}_{32}=0. \qquad (4.30)$$

Consequently, if the body has two mutually perpendicular planes of symmetry of elastic properties at each point, there are only nine non-zero elastic constants; they can be represented as the matrix

$$\begin{Vmatrix} C^{11}_{11} & C^{11}_{22} & C^{11}_{33} & 0 & 0 & 0 \\ C^{11}_{22} & C^{22}_{22} & C^{22}_{33} & 0 & 0 & 0 \\ C^{11}_{33} & C^{22}_{33} & C^{33}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C^{23}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{13}_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & C^{12}_{12} \end{Vmatrix}.$$

It follows from inspection of this matrix that if there are two orthogonal planes of elastic symmetry in a body, the third orthogonal plane is also a plane of elastic symmetry. Such a body is said to be orthotropic.

If the body possesses the same elastic properties with respect to each of the three planes of symmetry, and if the magnitude of the elastic potential is to remain unaltered when the axes ox_1, ox_2, ox_3 are permuted cyclically, i.e., when e_{11}, e_{22}, e_{33} or e_{12}, e_{23}, e_{31} are permuted cyclically, in addition to conditions (4.29) and (4.30) the following conditions must be fulfilled:

$$C^{11}_{11}=C^{22}_{22}=C^{33}_{33},\quad C^{22}_{11}=C^{33}_{11}=C^{33}_{22},\quad C^{12}_{12}=C^{13}_{13}=C^{23}_{23}. \qquad (4.31)$$

Consequently, when the body possesses the same elastic properties for each of the three planes, the elastic potential is of the form

$$A=\frac{C^{11}_{11}}{2}(e^2_{11}+e^2_{22}+e^2_{33})+C^{22}_{11}(e_{22}e_{33}+e_{33}e_{11}+e_{11}e_{22})+$$

$$+\frac{C^{12}_{12}}{2}(e^2_{12}+e^2_{23}+e^2_{31}) \qquad (4.32)$$

and only three independent elastic constants will remain.

Finally, if the body is isotropic, the elastic potential must be constant when any rotation of the co-ordinate axes is made. On the other hand, the stress tensor or the strain tensor has three independent invariants of the first, second, and third degree in the components of the stress and strain tensors. The elastic potential must therefore be expressed in terms of the invariants of the stress tensor if the elastic potential is represented by the components of the stress tensor, or in terms of the invariants of the strain tensor if the elastic potential is represented by the components of the strain tensor (4.28). Since the elastic potential is a homogeneous function of the second degree, it can contain only the first invariant to the second power and the second invariant to the first power, i.e.,

$$A = P\,(e_{11}+e_{22}+e_{33})^2+Q\,(e_{11}e_{22}+e_{22}e_{33}+e_{33}e_{11}-e^2_{12}-$$

$$-e^2_{23}-e^2_{31}). \qquad (4.33)$$

Thus, an isotropic body is characterized by only two elastic constants, P and Q. By applying formulas (4.20), and remembering that e^*_{kr} $(k \neq r)$ are the shearing strains in (4.20), whereas in (4.33) they denote half the shearing strains, from formula (4.33) we obtain

$$\sigma_{kr}=(2P+Q)\,\theta\delta_{kr}+(-Q)\,e_{kr},$$

where θ is the volume strain, δ_{kr} are the Kronecker symbols.

Taking into consideration the notation introduced by G. Lamé

$$2P+Q=\lambda, \quad -Q=2\mu, \qquad (4.34)$$

we obtain formulas for the components of the stress tensor for a linearly elastic isotropic body in terms of the components of the small strain tensor

$$\sigma_{kr}=\lambda\theta\delta_{kr}+2\mu e_{kr}. \qquad (4.35)$$

The constants λ and μ are called Lamé's elastic constants.

These formulas express the generalized Hooke's law for an isotropic body.

Note that, by virtue of formulas (4.34), formula (4.33) can now be represented as

$$2A=\lambda\theta^2+2\mu e_{rk}e_{kr}. \qquad (4.36)$$

Here both indices, k and r, are to be summed.

Since the strain energy is always positive, we conclude from formula (4.36) that $\mu > 0$. Indeed, if the tensor e_{kr} is chosen so that $e_{kk} = 0$, formula (4.36) takes the form

$$2A = 4\mu\,(e_{12}^2 + e_{23}^2 + e_{31}^2),$$

from which $\mu > 0$.

Let us prove that the principal directions of the strain tensor at each point of an isotropic body coincide with the principal direction of the stress tensor. By taking the principal directions of the strain tensor at some point of the body as the co-ordinate axes, we have $e_{12} = e_{23} = e_{31} = 0$; by formulas (4.35), $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$, which was to be proved. For isotropic bodies no distinction is therefore made between the principal directions of the strain tensor and those of the stress tensor; both are referred to as the principal directions.

Let an isotropic body be subjected to axial tensile loading; the state of stress at each of its points is given by

$$\sigma_{11} \neq 0, \quad \sigma_{22} = \sigma_{12} = \sigma_{23} = \sigma_{31} = \sigma_{33} = 0;$$

formulas (4.35) then become

$$\lambda\theta + 2\mu e_{11} = \sigma_{11}, \quad \lambda\theta + 2\mu e_{22} = 0, \ \lambda\theta + 2\mu e_{33} = 0. \tag{4.37}$$

By adding these formulas together, we obtain

$$\theta = \frac{1}{3\lambda + 2\mu}\,\sigma_{11}. \tag{4.38}$$

Inserting (4.38) in the first formula of (4.37), we have

$$\sigma_{11} = \frac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu}\,e_{11}. \tag{4.39}$$

On comparing (4.39) with the formula of Hooke's law for a prismatic rod in axial tension, we find

$$E = \frac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu}. \tag{4.40}$$

From the last two formulas of (4.37), with (4.38), we have

$$e_{22} = e_{33} = -\frac{\lambda}{2\mu}\,\theta = -\frac{\lambda}{2\mu\,(3\lambda + 2\mu)}\,\sigma_{11}.$$

Substitute the value of σ_{11} from (4.39) in this formula and introduce the notation

$$\nu = \frac{\lambda}{2\,(\lambda + \mu)}\,; \tag{4.41}$$

then

$$e_{22} = e_{33} = -\nu e_{11}. \tag{4.42}$$

Equalities (4.42) express the law of lateral contraction in axial tension; ν is called Poisson's ratio.

Suppose that an isotropic body is subjected to a uniform hydrostatic pressure; then

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \qquad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0. \tag{4.43}$$

By using (4.43), from (4.35) we obtain

$$\lambda\theta + 2\mu e_{rr} = -p \ (r = 1, 2, 3). \tag{4.44}$$

Adding formulas (4.44) together gives

$$(3\lambda + 2\mu)\,\theta = -3p. \tag{4.45}$$

By introducing the notation

$$K = \lambda + \frac{2}{3}\mu, \tag{4.46}$$

where K is known as the bulk modulus, from (4.45) we obtain

$$p = -K\theta. \tag{4.47}$$

According to the law of conservation of energy, when $p > 0$ there is a decrease in volume; taking this into account, from formula (4.47) we have $K > 0$. On the basis of formulas (4.40), (4.41), and (4.46) the quantities λ, μ, and K are expressed in terms of E and ν as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)}. \tag{4.48}$$

From the last two formulas of (4.48), remembering that $E > 0$, $\mu > 0$, and $K > 0$, we obtain $1 - 2\nu > 0$, $1 + \nu > 0$, from which the range of possible values of Poisson's ratio is

$$-1 < \nu < 0.5. \tag{4.49}$$

As seen, Poisson's ratio can also take some negative values. Experiments show, however, that Poisson's ratio for known materials takes positive values; instead of inequality (4.49) we therefore have

$$0 < \nu < 0.5.$$

For most materials ν has approximately the same value, close to 1/3. Consequently, from the first formula of (4.48) we have $\lambda > 0$.

Thus, it follows from (4.36) that the strain energy A is a positive definite quadratic form in the components of the strain tensor vanishing only when the components of the strain tensor are all zero simultaneously.

By solving (4.35) for the components of the strain tensor e_{kr}, and taking into account the first two formulas of (4.48), we obtain the generalized Hooke's law for an isotropic body:

$$e_{kr}=\frac{1+\nu}{E}\sigma_{kr}-\frac{3\nu}{E}\sigma\delta_{kr}, \tag{4.50}$$

where

$$\sigma=\frac{1}{3}\sigma_{kk}. \tag{4.51}$$

28. THERMAL STRESSES

Let T be a temperature change at some point of a body; the volume of a sufficiently small neighbourhood of this point will vary in proportion to T. In consequence, the extension of all fibres issuing from the given point is equal to αT. The components of the strain tensor are then

$$e'_{11}=e'_{22}=e'_{33}=\alpha T, \quad e'_{12}=e'_{23}=e'_{31}=0. \tag{4.52}$$

Here α is the coefficient of linear thermal expansion. Actually, the deformation of the neighbourhood of the point resulting from the temperature difference encounters environmental resistance. In this case the total strain e_{kr} is found by superimposing the above thermal expansion e'_{kr} and the elastic strain e''_{kr}, i.e.,

$$e''_{kr}=e_{kr}-e'_{kr}. \tag{4.53}$$

In an isotropic linearly elastic body, provided the proportional limit is not exceeded, the components of the strain tensor e''_{kr} are, by virtue of Neumann's hypothesis, related to the components of the stress tensor by the formulas of the generalized Hooke's law

$$\sigma_{kr}=\lambda\theta''\delta_{kr}+2\mu e''_{kr}. \tag{4.54}$$

According to formulas (4.52) and (4.53),

$$\theta''=\theta-3\alpha T. \tag{4.55}$$

Substituting (4.53) and (4.55) in formulas (4.54), we obtain

$$\sigma_{kr}=\lambda\theta\delta_{kr}+2\mu e_{kr}-\beta T\delta_{kr}. \tag{4.56}$$

Formula (4.56), where $\beta=(3\lambda+2\mu)\alpha$, expresses the generalized Hooke's law for an isotropic body. On the basis of Neumann's hypothesis the components of the total strain tensor appearing in formulas (4.56) are determined in terms of the displacements u_k by formulas (3.26).

29. ENERGY INTEGRAL FOR THE EQUATIONS OF MOTION OF AN ELASTIC BODY

Suppose that a body is acted on by a surface force T_n and a volume force ρF. Let us determine the work done by these forces from the initial moment $t=0$ corresponding to the natural state of rest to the moment under consideration t. The displacement of the points of the body during the time dt is $\frac{\partial u}{\partial t}\,dt$. Denote the work done by the external forces during the time dt by dR. Then

$$dR = \int_\tau \left(\rho F_k \frac{\partial u_k}{\partial t}\,dt\right) d\tau + \int_\omega \left(T_{nk} \frac{\partial u_k}{\partial t}\,dt\right) d\omega.$$

Substituting $T_{nk} = \sigma_{kr} n_r$, and transforming the surface integral into a volume one, we obtain

$$\frac{dR}{dt} = \int_\tau \left[\left(\frac{\partial \sigma_{kr}}{\partial x_r} + \rho F_k\right) \frac{\partial u_k}{\partial t} + \sigma_{kr} \frac{\partial}{\partial x_r}\left(\frac{\partial u_k}{\partial t}\right)\right] d\tau.$$

From the equations of motion we find

$$\frac{dR}{dt} = \int_\tau \rho \frac{\partial^2 u_k}{\partial t^2} \frac{\partial u_k}{\partial t}\,d\tau + \int_\tau \sigma_{kr} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial x_r}\right) d\tau.$$

Here the first term represents the derivative of the kinetic energy of the body with respect to time. Indeed,

$$\int_\tau \rho \frac{\partial^2 u_k}{\partial t^2} \frac{\partial u_k}{\partial t}\,d\tau = \frac{d}{dt} \int_\tau \frac{1}{2} \rho \left(\frac{\partial u_k}{\partial t}\right)^2 d\tau = \frac{dK}{dt}.$$

Because of the symmetry of the stress tensor σ_{kr},

$$\int_\tau \sigma_{kr} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial x_r}\right) d\tau = \int_\tau \sigma_{kr} \frac{\partial e_{kr}}{\partial t}\,d\tau.$$

If the process of elastic deformation takes place adiabatically or isothermally, we have, by (4.19),

$$\int_\tau \sigma_{kr} \frac{\partial e_{kr}}{\partial t}\,d\tau = \int_\tau \frac{\partial A}{\partial e_{kr}} \frac{\partial e_{kr}}{\partial t}\,d\tau = \frac{d}{dt} \int_\tau A\,d\tau.$$

Consequently,

$$\frac{dR}{dt} = \frac{d}{dt}\left(K + \int_\tau A\,d\tau\right).$$

By integrating the last relation between the limits 0 and t, and remembering that at the initial moment the body was in the natural state of rest, we find

$$R = K + \int_\tau A\,d\tau, \tag{4.57}$$

where $\int_\tau A\,d\tau$ is the work that must be expended by the external forces to produce deformation. This work is equal to the elastic strain energy; A is the elastic strain energy per unit volume. Formula (4.57) is true not only for isothermal and adiabatic processes, but A is then not a potential.

If the body is linearly elastic and isotropic, A is determined by formula (4.36). Thus, the work done by the external forces is expended in producing the kinetic energy of the body and the strain energy. Formula (4.57) expresses the law of conservation of mechanical energy.

If, under the action of the external forces, the body passes from the natural state of rest to a new, deformed, state of rest, the kinetic energy is zero and formula (4.57) takes the form

$$R=\int_{\tau} A\,d\tau. \tag{4.58}$$

30. BETTI'S IDENTITY

Let a linearly elastic body be in two different states of stress, $(\sigma'_{ij},\ e'_{ij})$ and $(\sigma''_{ij},\ e''_{ij})$. Then

$$\sigma'_{ij}=C_{ij}^{mn}e'_{mn}, \tag{4.59}$$

$$\sigma''_{ij}=C_{ij}^{mn}e''_{mn}. \tag{4.60}$$

By forming $\sigma'_{ij}e''_{ij}$, with (4.59), and grouping together the coefficients of e'_{ij}, with the use of (4.60), we obtain Betti's identity

$$\sigma'_{ij}e''_{ij}=\sigma''_{ij}e'_{ij}. \tag{4.61}$$

Betti's identity shows that for a linearly elastic body the work done by the first state of stress in the strain of the second state of stress is equal to the work done by the second state of stress in the strain of the first state of stress.

31. CLAPEYRON'S THEOREM

Let a linearly elastic body be in a state of rest under the action of a surface force T_n and a volume force ρF. The work done by the above forces during the displacements u_k is equal to

$$R=\int_{\tau}\rho F_k u_k\,d\tau+\int_{\omega}T_{nk}u_k\,d\omega.$$

Inserting $T_{nk}=\sigma_{kr}n_r$, and transforming the surface integral into a volume one, we obtain

$$R=\int_{\tau}\left[\left(\frac{\partial\sigma_{kr}}{\partial x_r}+\rho F_k\right)u_k+\sigma_{kr}\frac{\partial u_k}{\partial x_r}\right]d\tau.$$

By the equilibrium equations (2.25) and the symmetry of the stress tensor σ_{kr}, we have

$$R = \int_\tau \sigma_{kr} e_{kr} \, d\tau.$$

If the deformation process takes place adiabatically or isothermally we have, by Clapeyron's formula,

$$\int_\tau A \, d\tau = \frac{1}{2} R. \tag{4.62}$$

It follows from equality (4.62) that the elastic work of deformation is equal to half the work done by the statically applied external forces during the displacements. This proposition is known as Clapeyron's theorem.

CHAPTER V

Complete system of fundamental equations in the theory of elasticity

32. EQUATIONS OF ELASTIC EQUILIBRIUM AND MOTION IN TERMS OF DISPLACEMENTS

Equations (4.1) relating the stress and strain tensors complete the system of fundamental equations (2.24), (3.26) of the theory of elasticity, i.e., the resulting system of nine equations is

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho F_j = \rho \frac{\partial^2 u_j}{\partial t^2}, \tag{5.1}$$

$$\sigma_{ij} = f_{ij}(e_{rk}) \tag{5 2}$$

Here the unknowns are six components of the stress tensor σ_{ij} and three displacements u_k. The components of the small strain tensor e_{rk} are calculated in terms of u_k by means of formulas (3.26).

The system of equations (5.1) and (5.2) contains both the components of the displacement vector and the components of the stress tensor. In order to obtain the equations of equilibrium and motion in terms of displacements, from (5.2) we determine

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \frac{1}{2} \frac{\partial f_{ij}}{\partial e_{rk}} \left(\frac{\partial^2 u_r}{\partial x_i \, \partial x_k} + \frac{\partial^2 u_k}{\partial x_i \, \partial x_r} \right).$$

Here the coefficients of the second derivatives of the unknown functions u_k are functions of the first derivatives of these functions.

As a result, we have a system of three second-order non-linear partial differential equations in three functions u_k of three independent variables x_r in the case of equilibrium and four independent variables, x_r and the time t, in the case of dynamic application of forces. These equations are too complicated to be worth giving; it is more convenient to set them up directly in each particular problem.

In the case of a law of the form (4.6) the coefficients $\frac{\partial f_{ij}}{\partial e_{rk}}$ are the elastic constants C_{ij}^{rk}, and the differential equations constitute a system of three linear equations with variable coefficients when the body is non-homogeneous and with constant coefficients when the body is homogeneous.

For an isotropic homogeneous linearly elastic body we have

$$\frac{\partial\sigma_{ij}}{\partial x_i}=\lambda\delta_{ij}\frac{\partial\theta}{\partial x_i}+\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \qquad (5.3)$$

Substituting the expressions for these partial derivatives in Eqs. (5.1), we find

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_j}+\mu\Delta u_j+\rho\left(F_j-\frac{\partial^2 u_j}{\partial t^2}\right)=0. \qquad (5.4)$$

The resulting equations of motion in terms of displacements involving three functions u_j are called Lamé's differential equations. The system of equations (5.4) is equivalent to the differential equation in vector form

$$(\lambda+\mu)\nabla\operatorname{div}\boldsymbol{u}+\mu\Delta\boldsymbol{u}+\rho\left(\boldsymbol{F}-\frac{\partial^2\boldsymbol{u}}{\partial t^2}\right)=0, \qquad (5.5)$$

which is obtained from Eqs. (5.4) if each of them is multiplied by $\boldsymbol{i}_j$ and then summed with respect to the index j, remembering that $\theta=\operatorname{div}\boldsymbol{u}$. For elastic equilibrium, instead of the system of equations (5.4) and Eq. (5.5) we then have

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_j}+\mu\Delta u_j+\rho F_j=0, \qquad (5.6)$$

$$(\lambda+\mu)\nabla\operatorname{div}\boldsymbol{u}+\mu\Delta\boldsymbol{u}+\rho\boldsymbol{F}=0. \qquad (5.7)$$

In the case when the body is acted on by surface forces only, i.e., when the volume force $\rho\boldsymbol{F}$ is zero, Eqs. (5.6) take the form

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_j}+\mu\Delta u_j=0. \qquad (5.8)$$

Differentiating (5.8) with respect to x_j, and summing the index j, gives

$$(\lambda+2\mu)\Delta\theta=0,$$

from which

$$\Delta\theta=0.$$

Thus, in the absence of body forces the volume strain is a harmonic function.

By applying the operator Δ to both sides of (5.8), we obtain

$$(\lambda+\mu)\frac{\partial}{\partial x_j}\Delta\theta+\mu\Delta\Delta u_j=0.$$

Noting that $\Delta\theta=0$, we have

$$\Delta\Delta u_j=0,$$

i.e., the displacement vector is a biharmonic function.

Adding the formulas of the generalized Hooke's law (4.50) (when $r = k$) gives

$$\theta = 3\frac{1-2\nu}{E}\sigma, \tag{5.9}$$

where 3σ is the sum of the normal stresses acting on three mutually perpendicular planes. Since θ is a harmonic function, so also is σ by (5.9).

By applying the operator $\Delta\Delta$ to both sides of each of the formulas of the generalized Hooke's law for an isotropic and homogeneous body, and remembering that the volume strain is a harmonic function and u_j are biharmonic functions, we come to the conclusion that the stress components are also biharmonic functions.

Substituting the expression given by formula (4.56) in the differential equations (5.1), we obtain

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_j}+\mu\Delta u_j+\rho\left(F_j-\frac{\partial^2 u_j}{\partial t^2}\right)-\beta\frac{\partial T}{\partial x_j}=0. \tag{5.10}$$

These differential equations are called the Duhamel-Neumann thermoelastic equations.

It should be noted that the elastic constants λ and μ are functions of temperature T and, as established by experiments, they usually decrease with increasing temperature. In the case when temperature gradients in a body are not too great, λ and μ may be considered constant.

The system of differential equations (5.10) involves three unknown functions u_k since the temperature change is assumed to be known; the latter is determined as follows: let a body be subjected to a change in temperature depending on the co-ordinates of a point and the time t. Assume that the body is thermally isotropic and homogeneous; in addition, the thermal conductivity λ^* and the specific heat c are temperature independent. This assumption is fully justified when the temperature differences are not too great. In this case the function $T(x_1, x_2, x_3; t)$ must satisfy, throughout the body, the Fourier heat conduction equation

$$a^2\Delta T+\frac{Q}{c\gamma}=\frac{\partial T}{\partial t}, \tag{5.11}$$

where Δ is the Laplacian operator, γ is the specific weight, a is the thermal diffusivity, $a^2 = \lambda^* c^{-1}\gamma^{-1}$, Q is the quantity of heat per unit volume generated or absorbed per unit time by a heat source situated at the given point of the body.

In the case of a steady temperature state of a body Eq. (5.11) reduces to Poisson's equation

$$\Delta T = -\frac{Q}{\lambda^*}. \tag{5.12}$$

If the body contains no heat sources, we obtain Laplace's equation

$$\Delta T = 0, \tag{5.13}$$

i.e., the temperature is a harmonic function.

To determine the function T completely, it is necessary to specify the appropriate boundary conditions and also the initial conditions in the case of a transient temperature state. It is assumed that the unknown function and its partial derivatives are continuous up to the surface of the body. The initial value of T may be given by any, continuous or discontinuous, preassigned function $f(x_1, x_2, x_3)$, i.e.,

$$T(x_1, x_2, x_3;\quad 0) = f(x_1, x_2, x_3). \tag{5.14}$$

In the simplest case the boundary condition specifies the temperature over the surface ω of the body under consideration as a function of position and time

$$T\,|_\omega = T^\circ(x_1, x_2, x_3; t) \tag{5.15}$$

at any time $t > 0$.

On the boundary one can also prescribe the heat flow through the surface of the body

$$\left.\frac{\partial T}{\partial n}\right|_\omega = -q^\circ(x_1, x_2, x_3; t), \tag{5.16}$$

where q° is the heat flux entering or leaving the body across unit area of the body surface per unit time.

Finally, according to Newton's law of cooling, the condition on the surface of the body may also be given as

$$\left.\frac{\partial T}{\partial n}\right|_\omega = \frac{k}{\lambda^*}(T^* - T^\circ). \tag{5.17}$$

Here T^* is the temperature of the surrounding medium, k/λ^* is the relative heat transfer coefficient, k is the surface heat transfer coefficient.

In the case of a simultaneous consideration of the heat conduction problem and the thermoelastic problem we have to deal with the generalized heat conduction equation

$$\frac{\partial T}{\partial t} + b^2 \frac{\partial}{\partial t} \operatorname{div} \boldsymbol{u} = a^2 \Delta T. \tag{5.18}$$

Here $\boldsymbol{u}$ is the elastic displacement vector,

$$b^2 = \frac{c_p - c_v}{\alpha c_v} \quad (c_p > c_v),$$

c_p, c_v are the specific heats at constant pressure and constant volume, respectively; α is the coefficient of linear expansion.

The generalized heat conduction equation (5.18) differs from the usual equation (5.11) with $Q = 0$ by the presence of the additional term $b^2\frac{\partial}{\partial t}$ div u and can be obtained from the first law of thermodynamics. The idea of such a formulation of the problem is due to the well-known Russian physicist N. A. Umov who stated it in 1871 in *Theory of Thermomechanical Phenomena in Elastic Solids.* In this approach the solution of the thermoelastic problem is reduced to a simultaneous solution of the generalized heat conduction equation and the equations of motion and strain compatibility with the appropriate initial and boundary conditions for temperature and stresses. In this formulation the problem is realized when, in addition to temperature fields, the body is acted on by rapidly varying external forces, which may give rise to a rather significant redistribution of the temperature fields in the body, and this in turn may entail a redistribution of stresses. In cases where the thermal stresses in the body result only from external heating, the term $b^2\frac{\partial}{\partial t}$ div u can be neglected in the generalized heat conduction equation.

The heat conduction problem then becomes the first, and independent, step of the thermoelastic problem. Clearly, the term $b^2\frac{\partial}{\partial t}$ div u vanishes in all static problems, and hence here the thermoelastic problem and the heat conduction problem are solved separately.

By the foregoing method, from (2.30) and Hooke's law (4.35), using formulas (3.29) and (3.30), it is easy to obtain the differential equations of motion in terms of displacements in a cylindrical co-ordinate system. These are as follows:

$$\begin{aligned}
&(\lambda+2\mu)\, r\frac{\partial\theta}{\partial r}-2\mu\left(\frac{\partial\omega_3}{\partial\varphi}-\frac{\partial}{\partial x_3}(r\omega_\varphi)\right)+\rho r\left(F_r-\frac{\partial^2 u_r}{\partial t^2}\right)=0,\\
&(\lambda+2\mu)\frac{1}{r}\frac{\partial\theta}{\partial\varphi}-2\mu\left(\frac{\partial\omega_r}{\partial x_3}-\frac{\partial\omega_3}{\partial r}\right)+\rho\left(F_\varphi-\frac{\partial^2 u_\varphi}{\partial t^2}\right)=0, \qquad (5.19)\\
&(\lambda+2\mu)\, r\frac{\partial\theta}{\partial x_3}-2\mu\left(\frac{\partial}{\partial r}(r\omega_\varphi)-\frac{\partial\omega_r}{\partial\varphi}\right)+\rho r\left(F_3-\frac{\partial^2 u_3}{\partial t^2}\right)=0,
\end{aligned}$$

where

$$\theta=e_{rr}+e_{\varphi\varphi}+e_{33}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_3}{\partial x_3}. \qquad (5.20)$$

From (2.31) and (4.35), with the use of formulas (3.33) and (3.34), the differential equations of motion in terms of displacements in a spherical co-ordinate system are obtained as

$$\begin{aligned}
&(\lambda+2\mu)\, r\sin\psi\frac{\partial\theta}{\partial r}-\\
&-2\mu\left(\frac{\partial\omega_\psi}{\partial\varphi}-\frac{\partial}{\partial\psi}(\omega_\varphi\sin\psi)\right)+\rho r\sin\psi\left(F_r-\frac{\partial^2 u_r}{\partial t^2}\right)=0,
\end{aligned}$$

$$(\lambda+2\mu)\frac{1}{\sin\psi}\frac{\partial\theta}{\partial\varphi}-2\mu\left(\frac{\partial\omega_r}{\partial\psi}-\frac{\partial}{\partial r}(r\omega_\psi)\right)+\rho r\left(F_\varphi-\frac{\partial^2 u_\varphi}{\partial t^2}\right)=0,$$

$$(\lambda+2\mu)\sin\psi\frac{\partial\theta}{\partial\psi}-2\mu\left(\frac{\partial}{\partial r}(r\omega_\varphi\sin\psi)-\frac{\partial\omega_r}{\partial\varphi}\right)+$$
$$+\rho r\sin\psi\left(F_\psi-\frac{\partial^2 u_\psi}{\partial t^2}\right)=0, \tag{5.21}$$

where

$$\theta=e_{rr}+e_{\varphi\varphi}+e_{\psi\psi}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\psi}\left(\frac{\partial}{\partial\psi}(u_\psi\sin\psi)+\frac{\partial u_\varphi}{\partial\varphi}\right). \tag{5.22}$$

For an axially symmetric problem Eqs. (5.19) become

$$(\lambda+2\mu)r\frac{\partial\theta}{\partial r}+2\mu r\frac{\partial\omega_\varphi}{\partial x_3}+\rho r\left(F_r-\frac{\partial^2 u_r}{\partial t^2}\right)=0,\quad F_\varphi=0,$$
$$(\lambda+2\mu)r\frac{\partial\theta}{\partial x_3}-2\mu\frac{\partial}{\partial r}(r\omega_\varphi)+\rho r\left(F_3-\frac{\partial^2 u_3}{\partial t^2}\right)=0 \tag{5.23}$$

since $u_\varphi=0$, and u_r and u_3 are independent of the φ co-ordinate. Due to the last circumstance $\omega_r=\omega_3=0$ and

$$\theta=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_3}{\partial x_3}. \tag{5.24}$$

In the case of equilibrium and when $F_r=F_3=0$, Eqs. (5.23) reduce to

$$\frac{\partial\theta}{\partial r}+\frac{2\mu}{\lambda+2\mu}\frac{\partial\omega_\varphi}{\partial x_3}=0,$$
$$\frac{\partial\theta}{\partial x_3}-\frac{2\mu}{\lambda+2\mu}\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\varphi)=0. \tag{5.25}$$

From this

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\varphi)\right)+\frac{\partial^2\omega_\varphi}{\partial x_3^2}=0.$$

After determining the solution ω_φ of this equation, we find θ by using either of Eqs. (5.25). To determine u_r and u_3, from (3.30) and (5.24) we obtain differential equations of the form

$$\frac{\partial u_r}{\partial x_3}-\frac{\partial u_3}{\partial r}=2\omega_\varphi,$$
$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_3}{\partial x_3}=\theta. \tag{5.25'}$$

From this

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{\partial^2 u_r}{\partial x_3^2}=2\frac{\partial\omega_\varphi}{\partial x_3}+\frac{\partial\theta}{\partial r}.$$

After finding the solution u_r of the last equation, we also find u_3 by using either of Eqs. (5.25).

For an axially symmetric problem the equations of motion in a spherical co-ordinate system assume the form

$$(\lambda+2\mu)\frac{\partial\theta}{\partial r}+2\mu\frac{1}{r\sin\psi}\frac{\partial}{\partial\psi}(\omega_\varphi\sin\psi)+\rho r\left(F_r-\frac{\partial^2u_r}{\partial t^2}\right)=0,\quad F_\varphi=0,$$

$$(\lambda+2\mu)\frac{\partial\theta}{\partial\psi}-2\mu\frac{1}{\sin\psi}\frac{\partial}{\partial r}(r\omega_\varphi\sin\psi)+\rho r\left(F_\psi-\frac{\partial^2u_\psi}{\partial t^2}\right)=0,$$

where

$$\theta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\psi}\frac{\partial}{\partial\psi}(u_\psi\sin\psi),$$

$$\omega_\varphi=\frac{1}{2r}\left(\frac{\partial u_r}{\partial\varphi}-\frac{\partial}{\partial r}(ru_\psi)\right),\quad \omega_r=\omega_\psi=0$$

since $u_\varphi=0$, $u_\psi=u_\psi(r,\psi)$ and $u_r=u_r(r,\psi)$.

33. EQUATIONS IN TERMS OF STRESS COMPONENTS

Consider the basic equations of the statics of a linearly elastic isotropic body

$$\frac{\partial\sigma_{rk}}{\partial x_r}+\rho F_k=0, \tag{5.26}$$

$$e_{kr}=\frac{1+\nu}{E}\sigma_{kr}-\frac{3\nu}{E}\sigma\delta_{kr}. \tag{5.27}$$

Nine equations, (5.26) and (5.27), contain nine unknown functions u_k, σ_{kr}.

In Chap. III it has been stated that the six components of the strain tensor e_{kr} are not arbitrary functions of the co-ordinates of a point in a body, but must satisfy six Saint Venant's strain compatibility conditions. Remembering this, we substitute formulas (5.27) in Saint Venant's strain compatibility conditions; after some manipulation, we find six relations interconnecting the components of the stress tensor. Consequently, there are then in all three differential equations (5.26) and six relations among the components of the stress tensor, which we proceed to derive. Suppose that the body is homogeneous, i.e., λ and μ are independent of position. The resulting system of equations will apply only to isotropic homogeneous and linearly elastic bodies.

Substituting (5.27) in Saint Venant's conditions (3.45), after some rearrangement, we obtain

$$\frac{\partial^2\sigma_{11}}{\partial x_2^2}+\frac{\partial^2\sigma_{22}}{\partial x_1^2}-\frac{3\nu}{1+\nu}\left(\frac{\partial^2\sigma}{\partial x_1^2}+\frac{\partial^2\sigma}{\partial x_2^2}\right)=2\frac{\partial^2\sigma_{12}}{\partial x_1\,\partial x_2}, \tag{5.28}$$

$$\frac{\partial^2\sigma_{11}}{\partial x_2\,\partial x_3}-\frac{3\nu}{1+\nu}\frac{\partial^2\sigma}{\partial x_2\,\partial x_3}=\frac{\partial}{\partial x_1}\left(-\frac{\partial\sigma_{23}}{\partial x_1}+\frac{\partial\sigma_{31}}{\partial x_2}+\frac{\partial\sigma_{12}}{\partial x_3}\right) \tag{5.29}$$

and four more similar relations also corresponding to conditions (3.45). By differentiating the first equation of (5.26) with respect to x_1, the second with respect to x_2, the third with respect to $-x_3$, and adding, there results

$$-2\frac{\partial^2\sigma_{12}}{\partial x_1\,\partial x_2}=\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}-\frac{\partial^2\sigma_{33}}{\partial x_3^2}+\rho\left(\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}-\frac{\partial F_3}{\partial x_3}\right). \qquad (5.30)$$

Inserting (5.30) in (5.28), we have

$$\frac{\partial^2(\sigma_{11}+\sigma_{22})}{\partial x_1^2}+\frac{\partial^2(\sigma_{11}+\sigma_{22})}{\partial x_2^2}-\frac{\partial^2\sigma_{33}}{\partial x_3^2}-\frac{3\nu}{1+\nu}\left(\frac{\partial^2\sigma}{\partial x_1^2}+\frac{\partial^2\sigma}{\partial x_2^2}\right)=$$
$$=\rho\left(\frac{\partial F_3}{\partial x_3}-\frac{\partial F_2}{\partial x_2}-\frac{\partial F_1}{\partial x_1}\right)$$

or

$$\Delta\sigma-\frac{1}{3}(1+\nu)\Delta\sigma_{33}-\frac{\partial^2\sigma}{\partial x_3^2}=\frac{1}{3}(1+\nu)\rho\left(\frac{\partial F_3}{\partial x_3}-\frac{\partial F_2}{\partial x_2}-\frac{\partial F_1}{\partial x_1}\right),$$
$$\Delta\sigma-\frac{1}{3}(1+\nu)\Delta\sigma_{22}-\frac{\partial^2\sigma}{\partial x_2^2}=\frac{1}{3}(1+\nu)\rho\left(\frac{\partial F_2}{\partial x_2}-\frac{\partial F_1}{\partial x_1}-\frac{\partial F_3}{\partial x_3}\right), \qquad (5.31)$$
$$\Delta\sigma-\frac{1}{3}(1+\nu)\Delta\sigma_{11}-\frac{\partial^2\sigma}{\partial x_1^2}=\frac{1}{3}(1+\nu)\rho\left(\frac{\partial F_1}{\partial x_1}-\frac{\partial F_2}{\partial x_2}-\frac{\partial F_3}{\partial x_3}\right).$$

The last two relations are obtained in a similar way from the remaining two equations of the type (5.28).

By adding the last equalities, we find the formula

$$\Delta\sigma=-\rho\frac{1+\nu}{3(1-\nu)}\frac{\partial F_r}{\partial x_r}. \qquad (5.32)$$

Substituting (5.32) in (5.31), we obtain

$$\frac{1+\nu}{3}\Delta\sigma_{11}+\frac{\partial^2\sigma}{\partial x_1^2}=-\frac{2}{3}\rho(1+\nu)\frac{\partial F_1}{\partial x_1}-\rho\frac{\nu(1+\nu)}{3(1-\nu)}\frac{\partial F_r}{\partial x_r},$$
$$\frac{1+\nu}{3}\Delta\sigma_{22}+\frac{\partial^2\sigma}{\partial x_2^2}=-\frac{2}{3}\rho(1+\nu)\frac{\partial F_2}{\partial x_2}-\rho\frac{\nu(1+\nu)}{3(1-\nu)}\frac{\partial F_r}{\partial x_r}, \qquad (5.33)$$
$$\frac{1+\nu}{3}\Delta\sigma_{33}+\frac{\partial^2\sigma}{\partial x_3^2}=-\frac{2}{3}\rho(1+\nu)\frac{\partial F_3}{\partial x_3}-\rho\frac{\nu(1+\nu)}{3(1-\nu)}\frac{\partial F_r}{\partial x_r}.$$

These equalities constitute the first group of the Beltrami-Michell relations.

To obtain the second group of the Beltrami-Michell relations, we transform (5.29). For this purpose we differentiate the second equation of (5.26) with respect to x_3, the third with respect to x_2, and add them together; by adding the result to (5.29), we have

$$\frac{1+\nu}{3}\Delta\sigma_{23}+\frac{\partial^2\sigma}{\partial x_2\,\partial x_3}=-\rho\frac{1+\nu}{3}\left(\frac{\partial F_3}{\partial x_2}+\frac{\partial F_2}{\partial x_3}\right).$$

The remaining two equations of this type are obtained in a similar way.

Thus, the second group of relations takes the form

$$\begin{aligned}
&\frac{1+\nu}{3}\Delta\sigma_{12}+\frac{\partial^2\sigma}{\partial x_1\,\partial x_2}=-\rho\,\frac{1+\nu}{3}\left(\frac{\partial F_1}{\partial x_2}+\frac{\partial F_2}{\partial x_1}\right),\\
&\frac{1+\nu}{3}\Delta\sigma_{23}+\frac{\partial^2\sigma}{\partial x_2\,\partial x_3}=-\rho\,\frac{1+\nu}{3}\left(\frac{\partial F_2}{\partial x_3}+\frac{\partial F_3}{\partial x_2}\right),\\
&\frac{1+\nu}{3}\Delta\sigma_{31}+\frac{\partial^2\sigma}{\partial x_1\,\partial x_3}=-\rho\,\frac{1+\nu}{3}\left(\frac{\partial F_3}{\partial x_1}+\frac{\partial F_1}{\partial x_3}\right).
\end{aligned}\tag{5.34}$$

Consequently, the Beltrami-Michell relations represent six linear differential equations containing six functions σ_{rs}.

It is important to note that the system of equations (5.33) and (5.34) is suitable only for a linearly elastic isotropic homogeneous body in the case of isothermal or adiabatic deformation processes in the body, whereas six Saint Venant's compatibility equations are suitable for any body.

In the case when the body forces are absent or constant the Beltrami-Michell relations become

$$\frac{1+\nu}{3}\Delta\sigma_{kr}+\frac{\partial^2\sigma}{\partial x_k\,\partial x_r}=0.\tag{5.35}$$

Similarly, with the use of (4.50) and (2.30), when $F_r=F_\varphi=F_3==W_r=W_\varphi=W_3=0$, the compatibility equations (3.40) in a cylindrical co-ordinate system are rearranged in the form

$$\begin{aligned}
&\Delta\sigma_{rr}-\frac{2}{r^2}(\sigma_{rr}-\sigma_{\varphi\varphi})-\frac{4}{r^2}\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{1}{1+\nu}\frac{\partial^2\theta}{\partial r^2}=0,\\
&\Delta\sigma_{\varphi\varphi}+\frac{2}{r^2}(\sigma_{rr}-\sigma_{\varphi\varphi})+\frac{4}{r^2}\frac{\partial\sigma_{\varphi r}}{\partial\varphi}+\frac{1}{1+\nu}\frac{1}{r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial\varphi^2}\right)\theta=0,\\
&\Delta\sigma_{33}+\frac{1}{1+\nu}\frac{\partial^2\theta}{\partial x_3^2}=0,\\
&\Delta\sigma_{r\varphi}+\frac{1}{1+\nu}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\theta}{\partial\varphi}\right)+\frac{2}{r^2}\frac{\partial}{\partial\varphi}(\sigma_{rr}-\sigma_{\varphi\varphi})-\frac{4}{r^2}\sigma_{r\varphi}=0,\\
&\Delta\sigma_{\varphi3}+\frac{1}{1+\nu}\frac{1}{r}\frac{\partial^2\theta}{\partial\varphi\,\partial x_3}+\frac{2}{r^2}\frac{\partial\sigma_{3r}}{\partial\varphi}-\frac{\sigma_{\varphi3}}{r^2}=0,\\
&\Delta\sigma_{3r}+\frac{1}{1+\nu}\frac{\partial^2\theta}{\partial r\,\partial x_3}-\frac{2}{r^2}\frac{\partial\sigma_{3\varphi}}{\partial\varphi}-\frac{\sigma_{3r}}{r^2}=0,
\end{aligned}\tag{5.36}$$

where

$$\theta=\sigma_{rr}+\sigma_{\varphi\varphi}+\sigma_{33},\quad \Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial x_3^2}.$$

For an axially symmetric problem Eqs. (5.36) assume the form

$$\begin{aligned}
&\Delta\sigma_{rr}-\frac{2}{r^2}(\sigma_{rr}-\sigma_{\varphi\varphi})+\frac{1}{1+\nu}\frac{\partial^2\theta}{\partial r^2}=0,\\
&\Delta\sigma_{\varphi\varphi}+\frac{2}{r^2}(\sigma_{rr}-\sigma_{\varphi\varphi})+\frac{1}{1+\nu}\frac{1}{r}\frac{\partial\theta}{\partial r}=0,
\end{aligned}$$

$$\Delta\sigma_{33}+\frac{1}{1+\nu}\frac{\partial^2\theta}{\partial x_3^2}=0, \qquad (5.37)$$

$$\Delta\sigma_{3r}+\frac{1}{1+\nu}\frac{\partial^2\theta}{\partial r\,\partial x_3}-\frac{1}{r^2}\sigma_{3r}=0,$$

where

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x_3^2}$$

since $\sigma_{r\varphi}=\sigma_{\varphi 3}=0$, and the remaining components of the stress tensor are independent of the φ co-ordinate.

34. FUNDAMENTAL BOUNDARY VALUE PROBLEMS IN ELASTOSTATICS. UNIQUENESS OF SOLUTION

In practice the most common types of loading and fixing of bodies are the following: (1) the forces applied to the surface of a body are given; (2) the displacements are given at all points of the surface; (3) the displacements are given over a part of the surface, and the external forces over the remainder. In this connection three types of fundamental boundary value problems are distinguished in elastostatics.

The first fundamental boundary value problem consists in finding, in the region occupied by the body, three projections of the displacement vector and six components of the stress tensor, which must be continuous functions of position up to the surface of the body and satisfy Eqs. (5.1) and (5.2), and, in addition, the following conditions on its surface:

$$\sigma_{sr}n_r = T_{ns}(x_1, x_2, x_3), \qquad (5.38)$$

where T_{ns} are the projections of the given forces acting on the surface of the body.

The second fundamental boundary value problem consists in finding a solution of Eqs. (5.1) and (5.2) satisfying the following boundary conditions on the surface of the body:

$$u_r = u_r^\circ(x_1, x_2, x_3), \qquad (5.39)$$

where u_r° are the projections of the given displacement vector of the points of the body surface.

The third fundamental boundary value problem consists in determining a solution of Eqs. (5.1) and (5.2) satisfying conditions (5.38) over a part of the surface, and conditions (5.39) over the remainder. Besides these problems, other problems are encountered which also have applied significance. Some of these will subsequently come under consideration.

For thermoelastic problems involving surface forces as well as temperatures, the boundary conditions (2.22) take the form

$$\begin{aligned} T_{n1} &= (\lambda\theta + 2\mu e_{11} - \beta T)\, n_1 + 2\mu e_{12} n_2 + 2\mu e_{13} n_3, \\ T_{n2} &= 2\mu e_{21} n_1 + (\lambda\theta + 2\mu e_{22} - \beta T)\, n_2 + 2\mu e_{23} n_3, \\ T_{n3} &= 2\mu e_{31} n_1 + 2\mu e_{32} n_2 + (\lambda\theta + 2\mu e_{33} - \beta T)\, n_3. \end{aligned} \tag{5.40}$$

Equations (5.10) and (5.40) show that the elastic displacement vector $\boldsymbol{u}$ in the body is the same as that arising when the body is acted on by the forces $\beta\nabla T$ applied at each of its points and calculated per unit volume, and the pressure $\boldsymbol{n}\beta T$ on the surface, as well as the body and surface forces.

The proof of the existence of a solution of the above problems involves great mathematical difficulties. At present, however, the solvability of all boundary value problems of the theory of elasticity is established under rather general conditions. Assuming the existence of solutions of the foregoing boundary value problems, we proceed to the proof of their uniqueness.

Suppose that one of the above fundamental boundary value problems has two solutions, u'_k, σ'_{rk} and u''_k, σ''_{rk}. Obviously, the difference of these solutions

$$u_k = u'_k - u''_k, \qquad \sigma_{rk} = \sigma'_{rk} - \sigma''_{rk},$$

in the absence of body forces, must satisfy the basic elastostatic equations (5.1) and (5.2). Hence, formula (4.62) holds for u_k and σ_{rk}:

$$\int_{\omega} T_{nr} u_r \, d\omega = 2 \int_{\tau} A \, d\tau. \tag{5.41}$$

In the case of the first fundamental boundary value problem $T_{nr} = 0$ on the surface of the body for the solution made up of the difference of two solutions of the given problem since both solutions must satisfy conditions (5.38) for the same forces prescribed on the surface of this body.

In the case of the second fundamental boundary value problem, for the solution made up of the difference of two solutions of the given problem, we have $u_r = 0$ on the surface of the body, similarly to the preceding case.

Finally, in the case of the mixed problem $T_{nr} = 0$ over a part of the surface, and $u_r = 0$ over the remainder.

Thus, in all three fundamental boundary value problems the integrand is zero on the surface of the body, i.e.,

$$T_{nr} u_r = 0;$$

hence, in all three cases

$$\int_{\tau} A \, d\tau = 0. \tag{5.42}$$

Remembering that A is a positive quadratic form, from (5.42) we obtain

$$A = 0.$$

As follows from (4.36), this in turn is possible when $e_{rk} = 0$. We conclude from this that $e'_{rk} \equiv e''_{rk}$ or, on the basis of the generalized Hooke's law (4.35), $\sigma'_{rk} \equiv \sigma''_{rk}$. Consequently, both solutions give the same state of stress and strain.

Thus, the theorems of uniqueness of solution for the above problems are proved. It should be noted that it does not follow from the zero strain components, as may be inferred from formula (3.26), that $u_r \equiv 0$. In the solution of the first fundamental boundary value problem we can therefore obtain, for the projection of the displacement u_r, various values differing from one another only by a rigid-body displacement of the whole body, which has no effect on the state of stress or strain in the body. In the second and third fundamental boundary value problems there is no such difference because the displacements are given over the entire surface in the second problem or over a part of the surface in the third problem.

In this section we have proved that the system (5.1), (5.2) with given external forces uniquely determines the state of stress or strain in the body. In the foregoing proof of the uniqueness of solutions of the above-mentioned boundary value problems, which is given by G. Kirchhoff, the body may be assumed both simply connected and multiply connected.

35. FUNDAMENTAL PROBLEMS IN ELASTODYNAMICS

In the case of elastodynamics, as in statics, three fundamental problems may be formulated for Eqs. (5.4). In contrast to the fundamental boundary value problems in elastostatics, in the case of dynamic loading to the boundary conditions must be added the initial conditions specifying the projection of the displacement vector $\overset{\circ}{u}_k$ and the projection of the velocity vector $\overset{\circ}{v}_k$ of a point of the body at a certain time t_0 from which the study of the problem begins, i.e.,

$$u_k(x_1, x_2, x_3; t)\,|_{t=t_0} = \overset{\circ}{u}_k(x_1, x_2, x_3), \tag{5.43}$$

$$\frac{\partial}{\partial t} u_k(x_1, x_2, x_3; t)\,|_{t=t_0} = \overset{\circ}{v}_k(x_1, x_2, x_3). \tag{5.44}$$

Thus, the integrals of the system of equations (5.4) must satisfy not only the boundary conditions, but also the initial conditions (5.43), (5.44).

As in the preceding section, let us prove the uniqueness of solutions of the problems considered here, without taking up the proof of the existence theorems.

Assuming that these problems have two solutions, we consider their difference, which is the solution of the system of equations (5.4) with $F_k = 0$. For this solution, in the case of the first problem the stress vector on the surface of the body $T_n = 0$ for $t \geqslant t_0$; in the case of the second problem the displacement vector of a point of the surface of the body $u = 0$ for $t \geqslant t_0$, and hence $\frac{\partial u}{\partial t} = 0$ on the surface ω; in the case of the third problem $T_n = 0$ for $t \geqslant t_0$ over a part of the surface, and $u = 0$ for $t \geqslant t_0$ over the remainder; hence, $\frac{\partial u}{\partial t} = 0$ over this part of the surface of the body.

Since both solutions of the problem must satisfy the same initial conditions, it follows that the initial conditions for the difference of these solutions are homogeneous, i.e., at the initial moment t_0 we have

$$u = \frac{\partial u}{\partial t} = 0. \tag{5.45}$$

It appears from the above that the work R calculated for the difference of the solutions u for $t \geqslant t_0$ is zero. On this account, from formula (4.57) we have

$$K + \int_\tau A\, d\tau = 0. \tag{5.46}$$

Since the kinetic energy of the body K and the strain energy are positive quantities, from (5.46) we obtain

$$K = 0, \quad A = 0,$$

and hence

$$\frac{\partial u}{\partial t} = 0, \quad e_{rk} = 0 \quad \text{for} \quad t \geqslant t_0.$$

It follows from the condition $\frac{\partial u}{\partial t} = 0$ that the displacement vector u is time independent; it follows from the condition $e_{rk} = 0$ that the strain is zero. Consequently, the solution u can represent only a rigid-body displacement of the body. According to the condition of the problem, $u = 0$ at the initial moment; hence, this rigid-body displacement must be zero at all points of the body and at all times. Thus, the two solutions are completely coincident.

36. SAINT VENANT'S PRINCIPLE (PRINCIPLE OF SOFTENING OF BOUNDARY CONDITIONS)

Referring to the problems of bending and torsion of long prismatic bars, in 1855 B. de Saint Venant published his famous principle: *The mode of application and distribution of forces over the ends of a* *prism is immaterial for the effects produced over the remaining length, so that it is always possible, to a sufficient degree of approximation, to replace the applied forces by statically equivalent forces having the same resultant moment and the same resultant vector.*

Thirty years later, in 1885 the first general formulation of this principle was given by J. Boussinesq: *A balanced system of external forces applied to an elastic body when all points of application of the forces of this system lie inside a given sphere produces negligibly small deformations at distances from the sphere sufficiently greater than its radius.*

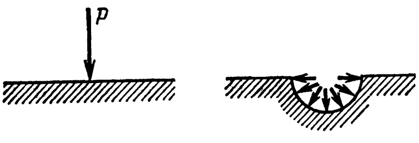

Fig. 12

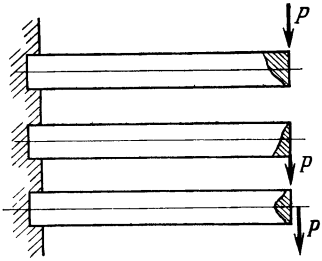

Fig. 13

To prove Saint Venant's principle, Boussinesq considered a semi-infinite body subjected to concentrated forces perpendicular to its plane boundary. It is rather interesting to note that up to now there has been no rigorously general proof of Saint Venant's principle. The existing attempts in this direction are primarily devoted to the estimation of the error of Saint Venant's principle as applied to prismatic bodies and also to bodies whose dimensions are all of the same order of magnitude. The problem of estimating the error of this principle in relation to thin-walled bars and shells is only slightly worked out on account of its great complexity.

When solving problems of the theory of elasticity reference is often made to Saint Venant's principle. If, in solving the problem, the boundary conditions are prescribed in full accord with the actual distribution of forces, the solution may be very complicated. Based on Saint Venant's principle, it is possible, by softening the boundary conditions, to obtain a solution which will give, for a major portion of the body, a field of the stress tensor very close to the actual one. The determination of the stress tensor in the region of load application presents special problems of the theory of elasticity called contact problems or problems of analysis of local stresses. Figure 12 shows two statically equivalent force systems: one in the form of a concentrated force ***P*** perpendicular to the plane boundary of a semi-infinite plate, and the other in the form of uniformly distributed forces over a semicylindrical surface whose resultant is equal to the force *P* and perpendicular to the boundary of the plate. At points sufficiently far removed from the region of application of the above forces the stress tensors in both cases are practically the same. The regions of a cantilever beam in which the stress tensor is essentially dependent on the mode of application of the force are dashed in Fig. 13.

The efficient solution of the boundary value problems of elastic equilibrium mentioned in Sec. 34 involves great difficulties in the general case. In this respect Saint Venant's principle holds a special place in the theory of elasticity. Owing to this principle, we have at present solutions of numerous problems of the theory of elasticity for Saint Venant's principle allows the boundary conditions to be softened; the given force system applied to a small part of an elastic body is replaced by any convenient (in simplifying the problem) statically equivalent force system applied to the same part of the surface of the body.

37. DIRECT AND INVERSE SOLUTIONS OF ELASTICITY PROBLEMS. SAINT VENANT'S SEMI-INVERSE METHOD

In direct solutions of problems dealing with elastic bodies we seek the stress and strain tensors and the displacement vector produced by the external forces acting on them. For this it is necessary to integrate Lamé's differential equations (5.4) if the fundamental unknowns are taken to be displacements u_k, or the differential equations (5.26) and the Beltrami-Michell relations (5.33), (5.34) if the fundamental unknowns are taken to be the components of the stress tensor with given boundary and initial conditions. In the first case it is said that the problem is solved in terms of displacements, and in the second case in terms of stresses.

In solutions of inverse problems we assign either displacements or the components of the strain tensor in the body under consideration and determine all the other quantities, including the external forces. The solutions of inverse problems present no great difficulties, but it is not always possible to arrive at solutions of any practical interest. Based on this, Saint Venant proposed a semi-inverse method consisting in partially specifying displacements and stresses simultaneously and then using the equations of the theory of elasticity to determine the equations that must be satisfied by the remaining displacements and stresses. The resulting equations are rather easily integrated. Thus, this method provides a complete and accurate solution for a large number of special problem most commonly encountered in practice. Saint Venant applied his method to the problems of unconstrained torsion and bending of prismatic bodies.

38. SIMPLE PROBLEMS OF THE THEORY OF ELASTICITY

The simple problems of the theory of elasticity will be defined as those in which the components of stress, and hence of strain, at any point of a body are constant or depend linearly on the co-ordinates. Obviously, in the simple problems the Beltrami-Michell relations or the strain continuity equations are satisfied identically. These problems are solved by the semi-inverse method.

1. All-round uniform pressure.

Let a body be subjected to an all-round uniform external pressure—**np** (**n** is the unit normal vector to the surface of the body). Body forces are neglected. Assign a stress tensor in the form

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0, \tag{5.47}$$

which satisfies the differential equations of equilibrium (2.25) and the Beltrami-Michell relations.

Determine the external forces producing a stress tensor of the form of (5.47) in the body under consideration; on the basis of formulas (2.22) we have

$$T_{nk} = -pn$$

or

$$\mathbf{T}_n = -\mathbf{n}p,$$

i.e., the stress vector applied on the surface of the body must represent the pressure $-\mathbf{n}p$, which is actually the case. Thus any three mutually perpendicular sections are principal planes at all points of the body. By the generalized Hooke's law (4.50), the components of the strain tensor are

$$e_{kk} = -\frac{1-2\nu}{E}p, \quad e_{kr} = 0 \quad (k \neq r). \tag{5.48}$$

Suppose that the given body is fixed at some point, which is taken as the origin of co-ordinate axes, and an elementary fibre situated on the x_3 axis is fixed at that point; moreover, the rotation of an elemetary fibre situated on the x_2 axis is constrained in the x_1x_2 plane. Analytically these fixing conditions are written respectively for all $x_k = 0$ as follows:

$$u_k = 0, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = \frac{\partial u_1}{\partial x_2} = 0. \tag{5.49}$$

Substituting solution (5.48) in formulas (3.44), and taking into account the fixing conditions (5.49), we determine, after simple computations, the projections of the elastic displacement vector

$$u_k = -\frac{1-2\nu}{E}px_k. \tag{5.50}$$

Consequently, the displacements of points of the body are radial; they increase directly with the distance from the origin and are symmetrical about it.

2. Axial extension of a prismatic rod.

Let a prismatic body with straight axis and bases of arbitrary shape at right angles to it be subjected to axial tension. Body forces are neglected. A stress tensor chosen in the form

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0, \quad \sigma_{33} = p \tag{5.51}$$

satisfies the differential equations of equilibrium (2.25) and the Beltrami-Michell relations. Determine the external forces producing a stress tensor of the form of (5.51) in the given prismatic body. By formulas (2.22) we have: on the lateral surface of the rod ($n_3 = 0$)

$$T_{n1} = T_{n2} = T_{n3} = 0, \tag{5.52}$$

on the bases of the rod [$\cos(x_3, x_1) = \cos(x_3, x_2) = 0$, $\cos(x_3, x_3) = \pm 1$]

$$T_{31} = T_{32} = 0, \quad T_{33} = +p. \tag{5.53}$$

Equalities (5.52) show that the lateral surface of the body must be free from external forces, which is exactly true since the body is acted on by axial forces only. Equalities (5.53) show that uniformly distributed tensile forces of intensity p must be applied to the bases of the rod. Actually the transmission of a tensile force to the rod under consideration may differ greatly from uniformly distributed tensile forces. According to Saint Venant's principle, however, solution (5.51) may be considered as exact over a part of the rod sufficiently far removed from its bases.

By the generalized Hooke's law (4.50), the components of the strain tensor are

$$\begin{gathered} e_{12} = e_{23} = e_{31} = 0, \\ e_{11} = -\frac{\nu}{E}\sigma_{33}, \quad e_{22} = -\frac{\nu}{E}\sigma_{33}, \quad e_{33} = \frac{\sigma_{33}}{E}. \end{gathered} \tag{5.54}$$

As in the first problem, at the centroid of the upper base of the rod, where the origin is placed, we assume the boundary conditions

$$u_k = 0, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_3} = \frac{\partial u_1}{\partial x_3} = 0 \quad \text{for} \quad x_k = 0. \tag{5.55}$$

Substituting solution (5.44) in formulas (3.44), with (5.55), we obtain

$$u_1 = -\frac{\nu p}{E} x_1, \quad u_2 = -\frac{\nu p}{E} x_2, \quad u_3 = \frac{p}{E} x_3. \tag{5.56}$$

The first two formulas of (5.56) show that the displacements u_1 and u_2 at all cross sections are the same and proportional to the distance of a given point of a cross section to the axis of the rod. The third formula of (5.56) shows that plane sections remain plane after deformation. In a course in strength of materials the last result is taken as the starting assumption known as the hypothesis of plane sections.

3. Extension of a prismatic rod under its own weight.

(a) Let a prismatic rod of length l fixed at its upper end be subjected to tensile deformation under its own weight. Denote the density of the material by ρ. We choose the axes of a co-ordinate system so that the origin is at the centroid of the upper base perpendicular to the axis of the rod, and one of the axes of the system, say x_3, directed vertically downwards, coincides with the axis of the rod. The projections of the mass force are then

$$F_1 = F_2 = 0, \quad F_3 = g.$$

By using Saint Venant's semi-inverse method, we choose the components of the stress tensor in the form

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0,$$
$$\sigma_{33} = ax_3 + b. \tag{5.57}$$

Here the constants a, b are not yet known. The Beltrami-Michell relations are satisfied identically by these components of the stress tensor; the first two equations of equilibrium are also satisfied identically, and from the third equation we obtain

$$a = -\rho g.$$

Now we have

$$\sigma_{33} = -\rho g x_3 + b.$$

Taking into account formulas (5.57), and remembering that $n_3 = 0$, where n is the outward normal to the surface of the rod, for the projection of the force acting on the lateral surface of the rod we have, by (2.22),

$$T_{n1} = T_{n2} = T_{n3} = 0.$$

As seen from the last equalities, the lateral surface of the rod must be free from forces, which is actually the case since the rod is under its own weight only. On the lower base of the rod ($x_3 = l$) we have

$$n_1 = n_2 = 0, \quad n_3 = 1,$$

where n is the outward normal to the lower base. According to formulas (2.22)

$$T^*_{n1} = 0, \quad T^*_{n2} = 0, \quad T^*_{n3} = -\rho g l + b.$$

By the conditions of the problem, the lower base is free from forces, hence

$$T^*_{n3} = -\rho g l + b = 0,$$

from which

$$b = \rho g l.$$

We have, finally,

$$\sigma_{33} = \rho g\,(-x_3 + l). \tag{5.58}$$

Proceeding in a similar way, on the upper base of the rod ($x_3 = 0$) we have

$$T^{**}_{n1} = T^{**}_{n2} = 0, \quad T^{**}_{n3} = -\rho g l.$$

This corresponds to fixing conditions of the upper base such that there are only normal stresses uniformly distributed throughout the base. No such fixing is practicable, but by virtue of Saint Venant's principle solution (5.57) may be considered to be exact for every other mode of fixing.

By the generalized Hooke's law, the components of the strain tensor are

$$\begin{gathered} e_{11} = e_{22} = -\frac{\nu}{E}\rho g\,(-x_3 + l), \quad e_{33} = \frac{1}{E}\rho g\,(-x_3 + l), \\ e_{12} = e_{23} = e_{31} = 0. \end{gathered} \tag{5.59}$$

For the boundary conditions (5.55) of the second problem, with (5.59), from formulas (3.44) we have

$$\begin{gathered} u_1 = \frac{\nu g \rho}{E}\,x_1\,(x_3 - l), \quad u_2 = \frac{\nu g \rho}{E}\,x_2\,(x_3 - l), \\ u_3 = -\frac{g\rho}{2E}\,[x_3^2 + \nu\,(x_1^2 + x_2^2) - 2 l x_3]. \end{gathered} \tag{5.60}$$

As the third formula of (5.60) shows, the points lying on the axis of the rod are displaced only along this axis.

Since there are no shearing stresses, and hence no shears, at the cross sections of the rod, these sections remain normal to all fibres after deformation of the rod and, as the third formula of (5.60) shows, are distorted into paraboloids of revolution which are convex downwards.

(b) A prismatic rod of length l fixed at its upper end is under the action of its own weight and a force P applied to the free end in the direction of the axis of the rod. We place the origin of co-ordinates at the centroid of the upper section and choose one of the axes, x_3, along the axis of the rod in the downward direction. Based on Saint Venant's principle, we replace the force P by a statically equivalent load of intensity $p = P/\omega$ uniformly distributed over the lower base of the rod (ω is the area of the lower base perpendicular to the axis of the rod); by reason of the linearity of the problem, the solution is represented as the sum of the solutions of the second and third problems.

4. Torsion of a circular prismatic bar.

Let the extreme cross sections of a circular prismatic bar with axis ox_3 be acted on by couples whose moments are equal in magnitude but opposite in sense; in this case the bar is subjected to torsion (Fig. 14); the lateral surface of the bar is free from surface forces, and there are no body forces ($F_k = 0$).

The elementary solution of the problem in the theory of strength of materials is based on the assumption that the cross sections of

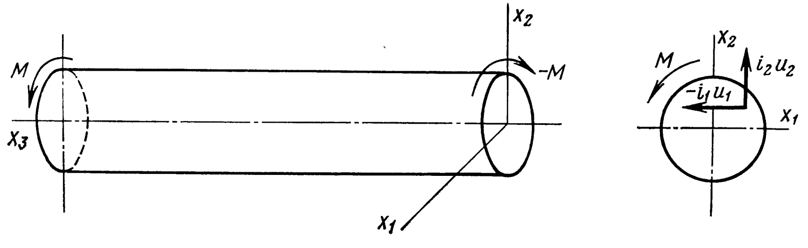

Fig. 14

the bar, remaining plane and at constant distances apart, rotate with respect to one another and their radii do not distort. If this assumption is taken into consideration, the projections of the displacement vector of some point in a certain cross section of the bar are

$$u_1 = -\tau x_2 x_3, \quad u_2 = \tau x_1 x_3, \quad u_3 = 0, \tag{5.61}$$

where τ is the constant angle of twist per unit length of the bar.

Let us examine whether these displacements are compatible with all the basic equations of the theory of elasticity. Substituting (5.61) in formulas (3.18), the components of the strain tensor are obtained as

$$\begin{gathered} e_{11} = e_{22} = e_{33} = e_{12} = 0, \\ 2e_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = \tau x_1, \\ 2e_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = -\tau x_2. \end{gathered} \tag{5.62}$$

The volume strain is

$$\theta = e_{11} + e_{22} + e_{33} = 0. \tag{5.63}$$

As seen from formulas (5.62), Saint Venant's compatibility equations are satisfied identically.

Noting that the mass forces are zero and relation (5.63) holds, we verify that displacements (5.61) satisfy Lame's equations of elastic equilibrium (5.8).

By the generalized Hooke's law, the stress components are equal to

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0, \qquad (5.64)$$

$$\sigma_{23} = \mu\tau x_1, \qquad \sigma_{31} = -\mu\tau x_2.$$

Thus, there are only two components of shearing stress acting at any cross section of the bar. Substituting (5.64) in formulas (2.22), we have $T_{nk} = 0$ on the lateral surface, where $n_3 = 0$. Consequently, the lateral surface of the bar must be free from stresses, which is actually the case. Further, substituting (5.64) in formulas (2.22) for the extreme cross sections ($n_1 = n_2 = 0$, $n_3 = \pm 1$), the surface forces corresponding to solution (5.61) are obtained as

$$T_{n1} = \mp \mu\tau x_2, \qquad T_{n2} = \pm \mu\tau x_1, \qquad T_{n3} = 0.$$

Thus, solution (5.61) leads to the conclusion that the extreme cross sections of the bar must be acted on by only tangential forces distributed according to law (5.64). The resultant vector and the resultant moment of these forces with respect to the centre of the circle are

$$V_1 = \int_\omega T_{n1}\, d\omega = -\mu\tau \int_\omega x_2\, d\omega,$$

$$V_2 = \int_\omega T_{n2} d\omega = \mu\tau \int_\omega x_1\, d\omega,$$

$$L = \int_\omega (T_{n2}x_1 - T_{n1}x_2)\, d\omega = \mu\tau \int_\omega (x_1^2 + x_2^2)\, d\omega.$$

Owing to the fact that the ox_1 and ox_2 axes pass through the centroid of the circle, the static moments of its area are

$$\int_\omega x_2\, d\omega = \int_\omega x_1\, d\omega = 0$$

We have, finally,

$$V_1 = V_2 = 0, \qquad L = \mu\tau I_0,$$

where I_0 is the polar moment of inertia of the area of the circle, i.e.,

$$I_0 = \int_\omega (x_1^2 + x_2^2)\, d\omega = \frac{\pi R^4}{2};$$

here R is the radius of the circumference.

The realization of the transmission of external forces at the ends of the bar according to law (5.64) is impracticable, but on the basis of Saint Venant's principle solution (5.64) may be considered exact for any law of transmission of external forces if the conditions of static equivalence are fulfilled, i.e., the constant τ is chosen so (this is possible) that the moment M of the applied couple at either of the extreme sections is equal to the resultant moment L_0:

$$L_0 = \mu\tau I_0 = M,$$

from which we obtain

$$\tau = \frac{M}{\mu I_0},$$

giving Hooke's law for a circular prismatic bar in torsion.

5. Pure bending of a prismatic bar.

Let the ox_3 axis be taken coincident with the axis of the bar, and the ox_1 and ox_2 axes coincident with the principal centroidal axes

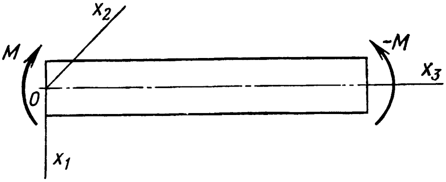

Fig. 15

of inertia of a cross section, the ox_1 axis being directed towards the stretched fibres (Fig. 15).

Suppose that the lateral surface of the bar is free from external forces, and that body forces are absent. Moreover, let the extreme sections of the bar under consideration be acted on by two couples whose planes of action coincide with one of its principal planes, the moments of the couples being equal in magnitude and opposite in sense. In this case the bar is subjected to pure bending and, as is known from the theory of strength of materials, the solution of this problem is based on the assumption that each cross section, remaining plane, rotates about a centroidal axis of this section perpendicular to the plane of action of the couples (the neutral axis ox_2) through a certain angle.

Assume that the components of the stress tensor are

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{32} = 0, \quad \sigma_{33} = ax_1, \tag{5.65}$$

where a is a constant, x_1 is the distance of the point of the cross section at which the normal stress σ_{33} is calculated to the neutral axis of this section.

Let us examine whether the stress components are compatible with the basic equations of the theory of elasticity. Since the problem under consideration is also a simple elasticity problem, the components of the stress tensor (5.65) identically satisfy the Beltrami-Michell relations. The components of the stress tensor (5.65) also satisfy the equations of elastic equilibrium.

On the lateral surface of the bar, where $n_3 = 0$, from formulas (2.22), with (5.65), we have

$$T_{n1} = T_{n2} = T_{n3} = 0.$$

Thus, the lateral surface must be free from external forces, which is actually the case.

At the extreme sections, where $\cos(x_3, x_1) = \cos(x_3, x_2) = 0$, $\cos(x_3, x_3) = \pm 1$, from formulas (2.22), with (5.65), we have

$$T^*_{n1} = T^*_{n2} = 0, \qquad T^*_{n3} = \pm ax_1. \tag{5.66}$$

Formulas (5.66) show that there must be only normal stresses distributed according to law (5.66) at the extreme sections of the bar. The resultant vector and the resultant moment of these forces are

$$V_3 = \int_\omega \sigma_{33}\,d\omega = a\int_\omega x_1\,d\omega,$$

$$L_1 = \int_\omega \sigma_{33}x_2\,d\omega = a\int_\omega x_1x_2\,d\omega,$$

$$L_2 = \int_\omega \sigma_{33}x_1\,d\omega = a\int_\omega x_1^2\,d\omega.$$

Taking into account that the ox_1 and ox_2 axes are directed along the principal centroidal axes of inertia of the cross section, the static moment with respect to the ox_2 axis and the product of inertia of the cross-sectional area with respect to the ox_1 and ox_2 axes are zero, we have, finally,

$$L_2 = aI,$$

where I is the moment of inertia of the cross-sectional area with respect to the neutral axis ox_2.

The transmission of external forces according to law (5.66) is impracticable; hence, based on Saint Venant's principle, instead of these forces one can take a load in the form of bending moments so that the equivalence condition is fulfilled, i.e.,

$$aI = M,$$

from which

$$a = \frac{M}{I},$$

and hence,

$$\sigma_{33} = \frac{M}{I}x_1.$$

By the generalized Hooke's law, the components of the strain tensor are equal to

$$e_{11} = e_{22} = -\frac{\nu}{E}\sigma_{33} = -\frac{\nu M}{EI}x_1,$$

$$e_{33} = \frac{M}{EI}x_1, \tag{5.67}$$

$$e_{12} = e_{23} = e_{31} = 0.$$

We place the origin at the centroid of the left extreme section and fix the bar at it so as to satisfy the conditions

$$u_1 = u_2 = u_3 = \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = \frac{\partial u_1}{\partial x_2} = 0.$$

From (3.44), using the fixing conditions and formulas (5.67), we obtain, after some manipulation, formulas for the determination of displacements:

$$u_1 = \frac{M}{2EI}\left[x_3^2 - \nu\left(x_1^2 - x_2^2\right)\right],$$

$$u_2 = -\frac{\nu M}{EI}x_1x_2, \tag{5.68}$$

$$u_3 = \frac{M}{EI}x_1x_3.$$

As formulas (5.68) show, the axis of the bar $x_1 = x_2 = 0$ before deformation, remaining in the ox_1x_3 plane called the plane of bending, is distorted into a parabola after deformation:

$$u_1 = -\frac{M}{2EI}x_3^2, \quad u_2 = u_3 = 0.$$

The curvature of the elastic line is, neglecting small quantities of higher order,

$$K = \frac{1}{R} = \frac{d^2u_1}{dx_3^2}.$$

Substituting the expression for u_1, we obtain the formula

$$\frac{1}{R} = -\frac{M}{EI},$$

which determines the curvature of the axis of the bar proportional to the magnitude of the bending moment. Since the curvature is constant, the elastic line, parabola, may be replaced by a circle, neglecting small quantities of higher order.

As follows from the formula for the displacement u_1, any cross section $x_3 = \text{constant}$ transforms into a plane section after deformation.

CHAPTER VI

The plane problem in the theory of elasticity

The solution of elasticity problems for the general case of three-dimensional bodies involves great mathematical difficulties; we are compelled by this circumstance to turn to the solution of more or less wide classes of special problems, one of which is the plane problem of elasticity. The latter comprises three cases of elastic equilibrium of a body having great practical significance, viz. plane strain, plane stress, and generalized plane stress.

39. PLANE STRAIN

The deformation of bodies is described as plane strain if the displacement vector of any point is parallel to a certain plane called the plane of deformation and is independent of the distance of the point under consideration to this plane.

Suppose that a body is subjected to plane strain parallel to the ox_1x_2 plane; then

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0. \tag{6.1}$$

Inserting (6.1) in formulas (3.26), the components of the strain tensor are obtained as

$$e_{11} = \frac{\partial u_1}{\partial x_1}, \quad e_{22} = \frac{\partial u_2}{\partial x_2}, \quad e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right); \tag{6.2}$$

the latter are in general different from zero and independent of x_3, and the remaining components are

$$e_{23} = e_{31} = e_{33} = 0.$$

The volume strain is then equal to

$$\theta_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$$

and is also a function only of the co-ordinates x_1 and x_2.

In this case the formulas of the generalized Hooke's law take the form

$$\sigma_{11} = \lambda\theta_1 + 2\mu e_{11}, \quad \sigma_{22} = \lambda\theta_1 + 2\mu e_{22},$$
$$\sigma_{12} = 2\mu e_{12}, \quad \sigma_{23} = \sigma_{31} = 0, \quad \sigma_{33} = \lambda\theta_1. \tag{6.3}$$

Consequently, in plane strain the stress tensor consists, in general, of four non-zero components depending on two arguments, x_1 and x_2. Because of the presence of the component σ_{33} a state of plane strain is achieved. It can easily be shown that for a body in plane strain the number of independent components of the stress tensor is three. Indeed, by adding the first two formulas of (6.3), and using the fifth formula of (6.3), we obtain

$$\sigma_{33}=\frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})$$

from which, with (4.41), we have

$$\sigma_{33}=\nu(\sigma_{11}+\sigma_{22}). \tag{6.4}$$

In the case considered the differential equations of equilibrium (2.25) become

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\rho F_1=0, \quad \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\rho F_2=0, \quad F_3=0. \tag{6.5}$$

These equations show that the mass force applied to any point of the body must be parallel to the plane of deformation and independent of the x_3 co-ordinate.

Lamé's equations (5.6) are also accordingly simplified and take the form

$$\begin{aligned}&(\lambda+\mu)\frac{\partial\theta_1}{\partial x_1}+\mu\Delta u_1+\rho F_1=0.\\&(\lambda+\mu)\frac{\partial\theta_1}{\partial x_2}+\mu\Delta u_2+\rho F_2=0;\end{aligned} \tag{6.6}$$

here Δ is the two-dimensional Laplacian operator.

Of Saint Venant's strain compatibility conditions, as is easily seen, there remains

$$\frac{\partial^2 e_{11}}{\partial x_2^2}+\frac{\partial^2 e_{22}}{\partial x_1^2}=2\frac{\partial^2 e_{12}}{\partial x_1\partial x_2}; \tag{6.7}$$

the other five conditions are satisfied identically.

For an isotropic homogeneous body the compatibility equation (6.7) in the absence of body forces becomes, by virtue of (6.3) and (6.5),

$$\Delta(\sigma_{11}+\sigma_{22})=0. \tag{6.8}$$

Indeed, from the formulas of Hooke's law (6.3), with (6.4), we have

$$\begin{aligned}e_{11}&=\frac{1}{2\mu}\left[\sigma_{11}-\frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})\right],\\e_{22}&=\frac{1}{2\mu}\left[\sigma_{22}-\frac{\lambda}{2(\lambda+\mu)}(\sigma_{21}+\sigma_{22})\right].\\e_{12}&=\frac{1}{2\mu}\sigma_{12};\end{aligned} \tag{6.9}$$

on the other hand, from the differential equations (6.5) with $F_1 = F_2 = 0$ we have

$$\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}=-2\frac{\partial^2\sigma_{12}}{\partial x_1\partial x_2}. \qquad (6.10)$$

Substituting (6.9) in (6.7), and using (6.10), we arrive at Eq. (6.8), which is called Lévy's equation.

Taking into account Hooke's law in the form of (6.9), the strain compatibility condition (6.7) may be given a new representation

$$\frac{\partial^2}{\partial x_2^2}[\sigma_{11}-\nu(\sigma_{11}+\sigma_{22})]+\frac{\partial^2}{\partial x_1^2}[\sigma_{22}-\nu(\sigma_{11}+\sigma_{22})]=2\frac{\partial^2\sigma_{12}}{\partial x_1\partial x_2}. \qquad (6.11)$$

From the definition of plane strain it follows that it is exactly realized in a prismatic body of infinite length with straight axis when the surface and body forces lie in the planes of cross sections and are independent of the co-ordinate along the axis of the body. When a prismatic body is of finite length, plane strain is not exactly realized in it. The longer the body, the more nearly does the deformation approach plane strain provided that the ends of the body are acted on by forces distributed according to the law $\sigma_{33} = \lambda\theta_1$.

Since, by definition, the conditions on the lateral surface of a prismatic body are independent of the x_3 co-ordinate, the boundary conditions are prescribed on the contour of one of the cross sections or on several contours if the section is multiply connected. Thus, the system of differential equations of equilibrium (6.5) and relations (6.3), together with contour conditions, describe simpler elastostatic problems (Sec. 34); here again, three fundamental two-dimensional boundary value problems may be distinguished.

According to (2.22), the contour conditions for the first fundamental boundary value problem are written as

$$\begin{aligned}\sigma_{11}n_1+\sigma_{12}n_2&=T_{n1},\\ \sigma_{12}n_1+\sigma_{22}n_2&=T_{n2}.\end{aligned} \qquad (6.12)$$

The differential equations of equilibrium and Lévy's equation as well as the contour conditions (6.12) in the absence of body forces contain no elastic constants of material. Consequently, in the case of plane strain in the absence of body forces the state of stress in the body at any simply connected section parallel to the plane of deformation is determined by the forces prescribed on the contour of this section, its shape, and is independent of the material properties.

If the section is a multiply connected region, the independence of the state of stress of the material properties is ensured by the additional condition that the external forces applied to each of the boundaries of the region should be balanced. (The proof of this proposition will be given somewhat later.) This statement constitutes the theorem of Maurice Lévy, which underlies the determination of the stress tensor on models from a material of different elastic properties.

In plane strain we obviously have

$$2\omega_{12}=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}, \quad 2\omega_{23}=0, \quad 2\omega_{13}=0.$$

For convenience, ω_{12} will be further denoted by ω_3. From the formulas

$$2\omega_3=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}, \qquad \theta=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2},$$

after determining the expressions

$$\Delta u_1=\frac{\partial \theta_1}{\partial x_1}-2\frac{\partial \omega_3}{\partial x_2}, \qquad \Delta u_2=\frac{\partial \theta_1}{\partial x_2}+2\frac{\partial \omega_3}{\partial x_1}$$

and substituting them in Lamé's equations of equilibrium (6.6), we obtain a system of differential equations for θ_1 and ω_3 in the form

$$(\lambda+2\mu)\frac{\partial \theta_1}{\partial x_1}-2\mu\frac{\partial \omega_3}{\partial x_2}+\rho F_1=0,$$

$$(\lambda+2\mu)\frac{\partial \theta_1}{\partial x_2}+2\mu\frac{\partial \omega_3}{\partial x_1}+\rho F_2=0.$$

Assuming $F_1=F_2=0$, the last equations become

$$\frac{\partial \theta_1}{\partial x_1}=\frac{\partial \omega^*}{\partial x_2}, \qquad \frac{\partial \theta_1}{\partial x_2}=-\frac{\partial \omega^*}{\partial x_1},$$

where $\omega^*=\frac{2\mu}{\lambda+2\mu}\omega_3$. These equations constitute the Cauchy-Riemann differential relations, and hence the functions θ_1, ω^* are conjugate harmonic functions.

40. PLANE STRESS

A state of stress in a plate is said to be plane if the stress vector on planes parallel to the bases is zero throughout its volume.

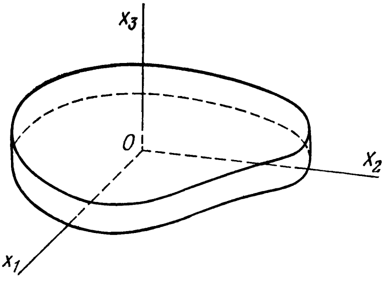

Fig. 16

Let the middle plane of the plate of thickness $2h$ be taken as the co-ordinate plane Ox_1x_2 (Fig. 16). By definition,

$$\sigma_{13}=\sigma_{23}=\sigma_{33}=0;$$

hence the system of differential equations (2.25) assumes the form of (6.5)

Since $\sigma_{33} = 0$, for an isotropic body we have

$$\sigma_{33} = \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + 2\mu\frac{\partial u_3}{\partial x_3} = 0.$$

Inserting, from this, the value of $\frac{\partial u_3}{\partial x_3}$ in terms of $\frac{\partial u_1}{\partial x_1}$ and $\frac{\partial u_2}{\partial x_2}$ in the remaining formulas of Hooke's law, we obtain relations between the components of the stress and strain tensors in the form

$$\begin{aligned} \sigma_{11} &= \lambda^*\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial u_1}{\partial x_1}, \\ \sigma_{22} &= \lambda^*\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial u_2}{\partial x_2}, \\ \sigma_{12} &= \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right). \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\frac{\partial u_3}{\partial x_3} = -\frac{\lambda}{\lambda+2\mu}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right), \\ &\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0, \quad \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0, \end{aligned} \tag{6.14}$$

where

$$\lambda^* = \frac{2\lambda\mu}{\lambda+2\mu}.$$

As seen, formulas (6.13) are obtained from the first three formulas of (6.3) by replacing Lamé's coefficient λ by the coefficient λ^*.

Substituting (6.14) in Eqs. (5.6), we obtain

$$(\lambda^* + \mu)\frac{\partial \theta_1}{\partial x_1} + \mu\Delta u_1 + \rho F_1 = 0,$$

$$(\lambda^* + \mu)\frac{\partial \theta_1}{\partial x_2} + \mu\Delta u_2 + \rho F_2 = 0, \quad F_3 = 0.$$

Here $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

These equations differ from Eqs. (6.6) only in that the coefficient λ is replaced by the coefficient λ^*, and are three-dimensional.

Thus, in spite of a considerable simplification in the basic equations for the plane stress problem, the problem remains three-dimensional since the x_3 co-ordinate is not eliminated from the foregoing equations. For the case when the plate thickness is sufficiently small, however, L. N. G. Filon propounded an idea permitting the reduction of the problem to a two-dimensional one. The idea is that the calculation of the means of the displacement vector and the stress tensor in a thin plate gives a reasonably accurate solution of the problem of plane stress; the latter, following A. E. H. Love, is termed 'generalized plane stress'.

41. GENERALIZED PLANE STRESS

Suppose now that a plate of height $2h$ is loaded on the lateral surface by external forces parallel to the bases and symmetrically distributed with respect to the middle plane; the bases of the plate are supposed to be free from external forces. Assume further that the component of the mass force perpendicular to the middle plane of the plate is zero, and that the other two components are disposed symmetrically with respect to the middle plane of the plate. The state of stress set up in such a plate is called generalized plane stress; it is often encountered in applications and is a practically important case.

By condition, on the bases $x_3 = \pm h$

$$T_{n1} = \sigma_{13} = 0, \quad T_{n2} = \sigma_{23} = 0, \quad T_{n3} = \sigma_{33} = 0, \tag{6.15}$$

and on the lateral surface of the plate $T_{n3} = 0$; also, $F_3 = 0$.

From the third differential equation of equilibrium

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho F_3 = 0,$$

using conditions (6.15), for $x_3 = \pm h$ we have

$$\frac{\partial \sigma_{33}}{\partial x_3} = 0.$$

Consequently, the derivative of σ_{33} with respect to the x_3 co-ordinate, as well as σ_{33}, vanishes when $x_3 = \pm h$; hence, if the plate thickness is sufficiently small, σ_{33} is very small, and we may assume $\sigma_{33} = 0$ throughout the plate.

It is obvious, by symmetry, that the projection of the displacement vector of any point of the middle plane on the ox_3 axis is zero and is an odd function in x_3; hence, its mean value is $u_3^* = 0$. We also assume that the variations of the projections $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$ across the thickness of the plate are small; instead of u_1, u_2 we may therefore consider their mean values across the thickness, which are determined by the formulas

$$u_1^* = \frac{1}{2h}\int_{-h}^{h} u_1\,dx_3, \quad u_2^* = \frac{1}{2h}\int_{-h}^{h} u_2\,dx_3. \tag{6.16}$$

Now multiply both sides of the system of differential equations of equilibrium (2.25) by $(2h)^{-1}dx_3$, and integrate with respect to the x_3 co-ordinate between the limits $-h$ and $+h$; then, by conditions

(6.15),

$$\frac{1}{2h}\int_{-h}^{h}\frac{\partial\sigma_{13}}{\partial x_3}\,dx_3=\frac{1}{2h}[\sigma_{13}]_{-h}^{h}=0,$$

$$\frac{1}{2h}\int_{-h}^{h}\frac{\partial\sigma_{23}}{\partial x_3}\,dx_3=\frac{1}{2h}[\sigma_{23}]_{-h}^{h}=0.$$

We finally obtain (remembering that $\sigma_{33}=0$, $F_3=0$)

$$\frac{\partial\sigma_{11}^*}{\partial x_1}+\frac{\partial\sigma_{12}^*}{\partial x_2}+\rho F_1^*=0,\quad \frac{\partial\sigma_{12}^*}{\partial x_1}+\frac{\partial\sigma_{22}^*}{\partial x_2}+\rho F_2^*=0, \tag{6.17}$$

$$\frac{\partial\sigma_{13}^*}{\partial x_1}+\frac{\partial\sigma_{23}^*}{\partial x_2}=0, \tag{6.18}$$

where

$$\sigma_{11}^*=\frac{1}{2h}\int_{-h}^{+h}\sigma_{11}\,dx_3,\ \ldots,\quad F_2^*=\frac{1}{2h}\int_{-h}^{+h}F_2\,dx_3$$

are the mean values of $\sigma_{11}, \ldots, F_2$ across the thickness of the plate.

It follows from the definition of plane stress that u_1, u_2, σ_{11}, σ_{22}, σ_{21} are even functions of x_3, and σ_{13}, σ_{23} are odd functions. Consequently, the mean values σ_{13}^*, σ_{23}^* are zero, and Eq. (6.18) is an identity.

By averaging the values of the given external forces on the lateral surface of the plate across its thickness on the contour of any section parallel to the bases (or on the contours if the section is multiply connected), we have

$$\begin{aligned}\sigma_{11}^*n_1+\sigma_{12}^*n_2&=T_{n1}^*,\\ \sigma_{12}^*n_1+\sigma_{22}^*n_2&=T_{n2}^*,\end{aligned} \tag{6.19}$$

where

$$T_{n1}^*=\frac{1}{2h}\int_{-h}^{+h}T_{n1}\,dx_3;\quad T_{n2}^*=\frac{1}{2h}\int_{-h}^{+h}T_{n2}\,dx_3.$$

Transforming to the mean values, and noting that $\sigma_{33}=0$, from the formulas of Hooke's law (6.13) we obtain

$$\begin{gathered}\sigma_{11}^*=\lambda^*\theta_1^*+2\mu\frac{\partial u_1^*}{\partial x_1},\quad \sigma_{22}^*=\lambda^*\theta_1^*+2\mu\frac{\partial u_2^*}{\partial x_2},\\ \sigma_{12}^*=\mu\left(\frac{\partial u_1^*}{\partial x_2}+\frac{\partial u_2^*}{\partial x_1}\right),\end{gathered} \tag{6.20}$$

where use has been made of the notation

$$\theta_1^*=\frac{\partial u_1^*}{\partial x_1}+\frac{\partial u_2^*}{\partial x_2}.$$

Relations (6.20) between the mean values of the components of the stress tensor σ_{11}^*, σ_{22}^*, σ_{12}^* and the derivatives of the mean values of the displacements u_1^*, u_2^* in generalized plane stress differ from relations (6.3) in plane strain only in that the constant λ^* takes the place of Lamé's elastic constant λ. The differential equations of equilibrium (6.17) and the contour conditions (6.19), which must be satisfied by σ_{11}^*, σ_{22}^*, σ_{12}^*, completely coincide with the differential equations of equilibrium (6.5) and the contour conditions (6.12) in plane strain. Consequently, for generalized plane stress Lamé's equilibrium equations and the Beltrami-Michell relations for the averaged values are written as in plane strain, (6.6) and (6.11), the only difference being that λ^* stands for λ.

Thus, we come to a very important conclusion that plane strain and generalized plane stress, being essentially different problems of plane elasticity, are mathematically identical.

42. AIRY'S STRESS FUNCTION

The solution of problems of plane elasticity is considerably simplified if body forces are disregarded either because of their smallness or remembering that a problem involving body forces can always be reduced to a problem with no body forces by finding some particular solution of the corresponding non-homogeneous differential equations of equilibrium. In the following discussion it will be assumed that there are no body forces.

In the plane problem of elasticity an auxiliary function first introduced by G. B. Airy plays an important part. It should be noted that owing to the introduction of this function an efficient method has been developed for the solution of problems of plane elasticity.

In the absence of body forces Eqs. (6.5) become

$$\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{12}}{\partial x_2}=0, \quad \frac{\partial \sigma_{12}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2}=0. \tag{6.21}$$

The first equation of (6.21) shows that the expression $\sigma_{11}\,dx_2 - \sigma_{12}\,dx_1$ is the total differential of a certain function $Q(x_1, x_2)$; hence,

$$\sigma_{12}=-\frac{\partial Q}{\partial x_1}, \quad \sigma_{22}=\frac{\partial P}{\partial x_2}. \tag{6.22}$$

Similarly, from the second equation we have

$$\sigma_{12}=-\frac{\partial P}{\partial x_2}, \quad \sigma_{22}=\frac{\partial Q}{\partial x_1}, \tag{6.23}$$

where $P(x_1, x_2)$ is some function. Comparison of these formulas for the same quantity gives the relation

$$\frac{\partial Q}{\partial x_1}=\frac{\partial P}{\partial x_2},$$

which shows that the expression $P\,dx_1+Q\,dx_2$ is the total differential of a certain function $\Phi(x_1, x_2)$, so that

$$P\,dx_1 + Q\,dx_2 = d\Phi,$$

whence

$$P=\frac{\partial\Phi}{\partial x_1}, \qquad Q=\frac{\partial\Phi}{\partial x_2}.$$

Substituting the values of P and Q in formulas (6.22) and (6.23), we obtain

$$\sigma_{11}=\frac{\partial^2\Phi}{\partial x_2^2}, \quad \sigma_{12}=-\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}, \quad \sigma_{22}=\frac{\partial^2\Phi}{\partial x_1^2}. \tag{6.24}$$

These formulas were first obtained by G. B. Airy. The function $\Phi(x_1, x_2)$ is called Airy's stress function.

Obviously, if it is assumed that relations (6.24) hold, Eqs. (6.21) are satisfied identically. Besides, as is known, for σ_{11}, σ_{22}, and σ_{12} to correspond to the actual state of stress, they must satisfy the compatibility conditions (6.8), i.e.,

$$\Delta(\sigma_{11}+\sigma_{22})=0. \tag{6.25}$$

On the other hand, from relations (6.24) we have

$$\sigma_{11}+\sigma_{22}=\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial x_2^2}=\Delta\Phi.$$

Taking into account the last equality, from (6.25) we obtain, finally,

$$\Delta\Delta\Phi=0, \tag{6.26}$$

where

$$\Delta\Delta=\frac{\partial^4}{\partial x_1^4}+2\frac{\partial^4}{\partial x_1^2\,\partial x_2^2}+\frac{\partial^4}{\partial x_2^4}.$$

In the following discussion it will be assumed that the stress function has continuous derivatives up to the fourth order in the region.

Thus, for the stress function to determine an actual state of stress, it is necessary and sufficient that it should be biharmonic.

Let us now derive contour conditions that must be satisfied by Airy's function. Assuming that the external forces T_{n1}, T_{n2} are given on the boundary of the region under consideration, we transform the contour conditions (6.12). It will be assumed in what follows that the contours are simple, i.e., not self-intersecting, and reasonably smooth.

We express n_1 and n_2 in terms of the derivatives of the co-ordinates x_1 and x_2 with respect to the arc length l measured in the positive sense along the contour under consideration. Referring to Fig. 17, we have

$$n_1=\frac{dx_2}{dl},\quad n_2=-\frac{dx_1}{dl}. \tag{6.27}$$

Inserting (6.24) and (6.27) in the contour conditions (6.12), we obtain, on L_r $(r=0,\ 1,\ 2,\ \ldots,\ m)$,

$$\frac{\partial^2\Phi}{\partial x_2^2}\frac{dx_2}{dl}+\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}\frac{dx_1}{dl}=T_{n1},$$

$$-\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}\frac{dx_2}{dl}-\frac{\partial^2\Phi}{\partial x_1^2}\frac{dx_1}{dl}=T_{n2}$$

or

$$\frac{d}{dl}\left(\frac{\partial\Phi}{\partial x_2}\right)=T_{n1},\ \frac{d}{dl}\left(\frac{\partial\Phi}{\partial x_1}\right)=-T_{n2}. \tag{6.28}$$

For an arbitrary point N of the contour L_r we introduce the notation

$$\left.\frac{\partial\Phi}{\partial x_1}\right|_N=A_r,\qquad \left.\frac{\partial\Phi}{\partial x_2}\right|_N=B_r \tag{6.29}$$

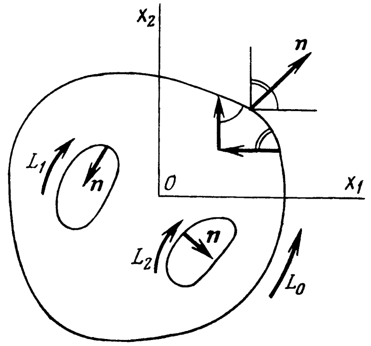

Fig. 17

By integrating equalities (6.28), we then obtain

$$\frac{\partial\Phi}{\partial x_1}=A_r-\int_N^M T_{n2}\,dl,\qquad \frac{\partial\Phi}{\partial x_2}=B_r+\int_N^M T_{n1}\,dl. \tag{6.30}$$

Thus, the increments of the functions $\frac{\partial\Phi}{\partial x_1}$, $\frac{\partial\Phi}{\partial x_2}$ in passing from N to M (these points lie on the same contour) are, respectively, equal to the projections on the ox_2 and ox_1 axes of the resultant vector of the external forces applied to the contour between these two points. With formulas (6.30), it is easy to find the derivatives

$$\frac{\partial\Phi}{\partial l}=\frac{\partial\Phi}{\partial x_1}\frac{dx_1}{dl}+\frac{\partial\Phi}{\partial x_2}\frac{dx_2}{dl}, \tag{6.31}$$

$$\frac{\partial\Phi}{\partial n}=\frac{\partial\Phi}{\partial x_1}\frac{dx_1}{dn}+\frac{\partial\Phi}{\partial x_2}\frac{dx_2}{dn}. \tag{6.32}$$

Inserting the values of $\frac{\partial\Phi}{\partial x_1}$, $\frac{\partial\Phi}{\partial x_2}$ from (6.30) in (6.31), and integrating the result thus obtained with respect to l, we have

$$\Phi=C_r+A_rx_1+B_rx_2+\int_N^M\left[-\frac{dx_1}{dl}\int_N^M T_{n2}\,dl+\frac{dx_2}{dl}\int_N^M T_{n1}\,dl\right]dl. \tag{6.33}$$

Formula (6.33) shows that if we are given the values of the external forces on any contour, the value of Φ can be calculated at any point of the same contour, apart from an additive linear expression of the form

$$C_r + A_r x_1 + B_r x_2.$$

It should be noted that this expression drops out for the components of the stress tensor calculated by formulas (6.24).

If the given region is simply connected, C_0, A_0, and B_0 may be taken equal to zero on the contour L_0. If the region is multiply connected, taking the constants C_r, B_r, and A_r to be zero on any one of the contours, we cannot choose the others arbitrarily.

Substituting the values of $\frac{\partial \Phi}{\partial x_1}$, $\frac{\partial \Phi}{\partial x_2}$ from (6.30) in (6.32), we determine the value of the normal derivative

$$\frac{\partial \Phi}{\partial n} = \left(A_r - \int_N^M T_{n2}\,dl\right)\frac{dx_1}{dn} + \left(B_r + \int_N^M T_{n1}\,dl\right)\frac{dx_2}{dn} \tag{6.34}$$

from the given external forces applied on the contours.

Thus, the solution of the plane problem of elasticity is reduced to the determination of a biharmonic function from the known contour values of this function and its normal derivative.

On passing once round any closed contour, we have, by formulas (6.30) and (6.33),

$$\begin{aligned} \frac{\partial \Phi}{\partial x_1} &= -\oint T_{n2}\,dl = -V_2, \\ \frac{\partial \Phi}{\partial x_2} &= \oint T_{n1}\,dl = V_1 \end{aligned} \tag{6.35}$$

$$\Phi = \oint \left[-\frac{dx_1}{dl} \int_N^M T_{n2}\,dl + \frac{dx_2}{dl} \int_N^M T_{n1}\,dl \right] dl, \tag{6.36}$$

where V_1 and V_2 are, respectively, the projections on the ox_1 and ox_2 axes of the resultant vector of the external forces applied to the contour under consideration. By integrating (6.36) by parts, we obtain

$$\Phi = -x_{1N}V_2 + x_{2N}V_1 + \oint (x_1 T_{n2} - x_2 T_{n1})\,dl, \tag{6.37}$$

where x_{1N} and x_{2N} are the co-ordinates of an arbitrary point N of the contour under consideration L_r from which a complete circuit is started. The third term in (6.37) determines the value of the resultant moment of all external forces applied to the given contour about an arbitrarily chosen origin of co-ordinates.

Formulas (6.35) and (6.37) enable one to establish conditions for single-valuedness of the function Φ and its derivatives $\frac{\partial\Phi}{\partial x_1}$, $\frac{\partial\Phi}{\partial x_2}$. The function Φ and its derivatives $\frac{\partial\Phi}{\partial x_1}$, $\frac{\partial\Phi}{\partial x_2}$ are single valued if the resultant vector and the resultant moment of the external forces applied to every contour of the region are each zero; if the resultant vector is zero, the function is not, in general, single valued, and its derivatives $\frac{\partial\Phi}{\partial x_1}$, $\frac{\partial\Phi}{\partial x_2}$ are single-valued functions; if, however, the resultant vector is not zero, both the function Φ itself and its derivatives are not single valued.

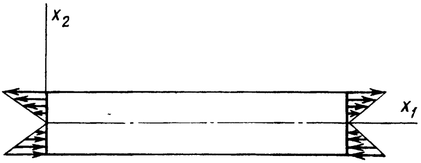

Fig. 18

A number of interesting solutions of Eq. (6.26) can be obtained by assigning Airy's function in the form of polynomials of different degrees. As the simplest example we choose Airy's function in the form of a second-degree polynomial, which obviously satisfies Eq. (6.26),

$$\Phi = \frac{1}{2} a_2 x_1^2 + b_2 x_1 x_2 + \frac{1}{2} c_2 x_2^2.$$

In the absence of body forces the components of the stress tensor are, from Airy's formulas (6.24),

$$\sigma_{11} = c_2, \quad \sigma_{22} = a_2, \quad \sigma_{12} = -b_2.$$

Thus, all three components are constant in the entire region. For a rectangular strip with sides parallel to the co-ordinate axes (Fig. 18), the forces applied to the contour where $\alpha_{11} = \pm 1$, $\alpha_{22} = \pm 1$ are, by formula (6.12),

$$T_{11} = \sigma_{11} = c_2, \quad T_{12} = \sigma_{12} = -b_2, \quad T_{22} = \sigma_{22} = a_2.$$

Equation (6.26) is also satisfied by a third-degree polynomial

$$\Phi = \frac{a_3}{6} x_1^3 + \frac{b_3}{2} x_1^2 x_2 + \frac{c_3}{2} x_1 x_2^2 + \frac{d_3}{6} x_2^3.$$

On the basis of formula (6.24) the stress components are

$$\begin{gathered} \sigma_{11} = c_3 x_1 + d_3 x_2, \quad \sigma_{22} = a_3 x_1 + b_3 x_2, \\ \sigma_{12} = -b_3 x_1 - c_3 x_2. \end{gathered} \tag{6.38}$$

Assuming $c_3 = a_3 = b_3 = 0$, $d_3 \neq 0$, we obtain

$$\sigma_{11} = d_3x_2, \quad \sigma_{22} = 0, \quad \sigma_{12} = 0. \tag{6.39}$$

This system of components of the stress tensor corresponds to pure bending of a rectangular strip by external forces applied at both its ends, $x_1 = 0$, $x_1 = l$. These external forces must be equal, by formulas (6.12), to $-d_3x_2$ at the end $x_1 = 0$ and to d_3x_2 at the end $x_1 = l$. The resultant vector and the resultant moment of these forces are obviously given by

$$V = \delta \int_{-c}^{c} \sigma_{11}\, dx_2 = 0, \quad M = \delta \int_{-c}^{c} \sigma_{11}x_2\, dx_2 = \frac{2}{3}\, d_3c^3\delta.$$

Here δ is the thickness of the strip, $2c$ is its depth.

By Saint Venant's principle, the solution found above is also applicable well away from the ends of the strip when, instead of the external forces applied at both ends of the strip and distributed according to law (6.39), there are statically equivalent couples of moment M, the state of stress differing from (6.39) near the region of application of the couples. If the only non-zero coefficient is a_3, the non-vanishing component of the stress tensor is the normal stress $\sigma_{22} = a_3x_1$. If, however, only one of the coefficients b_3, c_3 is different from zero, say $c_3 \neq 0$, there is a shearing stress σ_{12} in addition to the normal stress σ_{11}. When use is made of polynomials of higher degree than the third, the biharmonic equation is satisfied for certain relations between their coefficients.

43. AIRY'S FUNCTION IN POLAR CO-ORDINATES. LAMÉ'S PROBLEM

The equilibrium equations for the plane problem of elasticity in a polar co-ordinate system become, on the basis of Eq. (2.30) in the absence of body forces,

$$\begin{gathered} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \\ \frac{1}{r}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{2}{r}\sigma_{r\varphi} = 0. \end{gathered} \tag{6.40}$$

The solution of this system may be taken in the form

$$\begin{gathered} \sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2}, \quad \sigma_{\varphi\varphi} = \frac{\partial^2 \Phi}{\partial r^2}, \\ \sigma_{r\varphi} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \varphi} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r\, \partial \varphi}, \end{gathered} \tag{6.41}$$

where $\Phi(r, \varphi)$ is Airy's stress function in a polar co-ordinate system.

The first two relations of (6.41) must satisfy condition (6.25), i.e.,

$$\Delta(\sigma_{rr}+\sigma_{\varphi\varphi})=0, \tag{6.42}$$

where Δ is the two-dimensional Laplacian operator in a polar co-ordinate system

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}.$$

Substituting the expressions for σ_{rr}, $\sigma_{\varphi\varphi}$ from (6.41) in (6.42), we obtain a biharmonic equation for the determination of Airy's function:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2}\right)=0. \tag{6.43}$$

In the case of a symmetrical distribution of stress about the origin Eq. (6.43) takes the form

$$\frac{d^4\Phi}{dr^4}+\frac{2}{r}\frac{d^3\Phi}{dr^3}-\frac{1}{r^2}\frac{d^2\Phi}{dr^2}+\frac{1}{r^3}\frac{d\Phi}{dr}=0,$$

and the general solution is

$$\Phi=A\ln r+Br^2\ln r+Cr^2+D. \tag{6.44}$$

Substituting (6.44) in formulas (6.41), we obtain the components of the stress tensor in the case of a symmetrical distribution of stress about the origin:

$$\begin{aligned}\sigma_{rr}&=A\frac{1}{r^2}+B(1+2\ln r)+2C,\\ \sigma_{\varphi\varphi}&=-A\frac{1}{r^2}+B(3+2\ln r)+2C,\\ \sigma_{r\varphi}&=0.\end{aligned} \tag{6.45}$$

If the point $r=0$ belongs to the region, A and B must be taken equal to zero to make the components of the stress tensor bounded; then

$$\sigma_{rr}=\sigma_{\varphi\varphi}=C.$$

The problem of the deformation of a hollow circular cylinder subjected to a uniform pressure on the inner and outer surfaces was first solved by G. Lamé. The solution of this problem can easily be obtained from relations (6.45) subject to the boundary conditions

$$\begin{aligned}\sigma_{rr}&=-p_1 \text{ on the cylinder } r=r_1,\\ \sigma_{rr}&=-p_2 \text{ on the cylinder } r=r_2,\end{aligned} \tag{6.46}$$

where r_1, r_2 are, respectively, the inner and outer radii of the cylinder.

The determination of the coefficients A, B, C requires a third condition in addition to two boundary conditions (6.46). The third condition is the independence of the projections of the displacement vector u_r, u_φ of the polar angle φ since the independence of the components of the stress tensor of the angle φ does not necessarily lead to the independence of the displacement vector of the polar angle φ. In the case of plane strain u_r and u_φ are determined from the formulas of Hooke's law:

$$\frac{\partial u_r}{\partial r}=\frac{1+\nu}{E}\left[(1-\nu)\,\sigma_{rr}-\nu\sigma_{\varphi\varphi}\right]=$$
$$=\frac{1+\nu}{E}\left\{\frac{A}{r^2}+B\left[(1-4\nu)+2\,(1-2\nu)\ln r\right]+2C\,(1-2\nu)\right\},$$
$$\frac{u_r}{r}+\frac{1}{r}\,\frac{\partial u_\varphi}{\partial \varphi}=\frac{1+\nu}{E}\left[-\nu\sigma_{rr}+(1-\nu)\,\sigma_{\varphi\varphi}\right]=$$
$$=\frac{1+\nu}{E}\left\{-\frac{A}{r^2}+B\left[(3-4\nu)+2\,(1-2\nu)\ln r\right]+2C\,(1-2\nu)\right\}.$$

From these relations we find

$$u_r=\frac{1+\nu}{E}\left\{-\frac{A}{r}-B\left[1-2\,(1-2\nu)\ln r\right]r+2C\,(1-2\nu)\,r\right\}+f\,(\varphi),$$
$$u_\varphi=\frac{4\,(1-\nu^2)}{E}\,Br\varphi-\int f\,(\varphi)\,d\varphi+g\,(r). \tag{6.47}$$

Here the functions $f\,(\varphi)$ and $g\,(r)$ are to be determined.

Since the projections of the displacement vector u_r, u_φ must be independent of φ, we have to put

$$f\,(\varphi)=0, \quad B=0. \tag{6.48}$$

On putting $B = 0$ in relations (6.45), from the boundary conditions (6.46) we find

$$\frac{A}{r_1^2}+2C=-p_1, \quad \frac{A}{r_2^2}+2C=-p_2.$$

By solving this system of equations, we obtain

$$A=\frac{(p_2-p_1)\,r_1^2r_2^2}{r_2^2-r_1^2}, \quad C=\frac{r_1^2p_1-rp_2}{2\,(r_2^2-r_1^2)}. \tag{6.49}$$

Inserting (6.49) in (6.45), the stress components are, finally,

$$\sigma_{rr}=\frac{r_1^2r_2^2\,(p_2-p_1)}{r_2^2-r_1^2}\,\frac{1}{r^2}+\frac{r_1^2p_1-r_2^2p_2}{r_2^2-r_1^2},$$
$$\sigma_{\varphi\varphi}=-\frac{r_1^2r_2^2\,(p_2-p_1)}{r_2^2-r_1^2}\,\frac{1}{r_2}+\frac{r_1^2p_1-r_2^2p_2}{r_2^2-r_1^2}. \tag{6.50}$$

Substituting (6.49) in (6.47), we have

$$u_r=\frac{1+\nu}{E\,(r_2^2-r_1^2)}\left[r_1^2r_2^2\,(p_1-p_2)\,\frac{1}{r}+(1-2\nu)\,(r_1^2p_1-r_2^2p_2)\,r\right],$$
$$u_\varphi=g\,(r). \tag{6.51}$$

Since $\sigma_{r\varphi} = 0$, it follows that

$$2\mu e_{r\varphi} = 2\mu \frac{1}{2}\left(\frac{du_\varphi}{dr} - \frac{u_\varphi}{r}\right) = 0,$$

and then

$$\frac{dg(r)}{g(r)} = \frac{dr}{r},$$

from which

$$g(r) = qr,$$

where q is an integration constant.

Thus, the tangential projection of the displacement vector u_φ represents a rotation of an absolutely rigid body.

Let us now consider the problem of determining the state of stress in a thin concentric circular disk rotating with a constant angular velocity ω. The disk is acted on by the volume force $\rho F_r = \rho\omega^2 r$.

Noting that here the deformation is symmetrical about the pole O, we have $u_\varphi = 0$, $\frac{\partial u_r}{\partial \varphi} = 0$. Hence, from formulas (3.29) and (2.30),

$$e_{rr} = \frac{du_r}{dr}, \quad e_{\varphi\varphi} = \frac{u_r}{r}, \tag{6.52}$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} + \rho\omega^2 r = 0. \tag{6.53}$$

Taking into account (6.52), from (3.40) we obtain the strain compatibility equation in the form

$$\frac{1}{r}\frac{d}{dr}\left(r^2 \frac{de_{\varphi\varphi}}{dr}\right) - \frac{de_{rr}}{dr} = 0$$

or

$$\frac{d}{dr}\left[r\frac{de_{\varphi\varphi}}{dr} + (e_{\varphi\varphi} - e_{rr})\right] = 0.$$

The last relation is satisfied if

$$\frac{de_{\varphi\varphi}}{dr} = \frac{1}{r}(e_{rr} - e_{\varphi\varphi}). \tag{6.54}$$

By using the formulas of Hooke's law (5.27), and noting that $\sigma_{33} = 0$, $\sigma_{rr} = \sigma_{rr}(r)$, $\sigma_{\varphi\varphi} = \sigma_{\varphi\varphi}(r)$, we rearrange relation (6.54) in the form

$$\frac{d\sigma_{\varphi\varphi}}{dr} - \nu\frac{d\sigma_{rr}}{dr} = \frac{1+\nu}{r}(\sigma_{rr} - \sigma_{\varphi\varphi}).$$

Substituting in this equation the expression for $\sigma_{rr} - \sigma_{\varphi\varphi}$ from (6.53), we have

$$\frac{d\sigma_{\varphi\varphi}}{dr} = -\frac{d\sigma_{rr}}{dr} - (1+\nu)\rho\omega^2 r. \tag{6.55}$$

Differentiating (6.53) with respect to r gives

$$\frac{d}{dr}\left(r\frac{d\sigma_{rr}}{dr}\right)+\frac{d\sigma_{rr}}{dr}-\frac{d\sigma_{\varphi\varphi}}{dr}+2\rho\omega^2 r=0.$$

Substituting (6.55) in the last equation, we find

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\sigma_{rr})\right]+(3+\nu)\rho\omega^2 r=0.$$

Integrate this equation:

$$\sigma_{rr}=A+\frac{B}{r^2}-\frac{3+\nu}{8}\rho\omega^2 r^2. \tag{6.56}$$

From (6.53) and (6.56) we have

$$\sigma_{\varphi\varphi}=A-\frac{B}{r^2}-\frac{1+3\nu}{8}\rho\omega^2 r^2.$$

To find the constants A and B we have the following boundary conditions:

$$\sigma_{rr}=0 \text{ on the cylinder } r=r_2,$$
$$\sigma_{rr}=0 \text{ on the cylinder } r=r_1.$$

From this

$$A+\frac{B}{r_2^2}-\frac{3+\nu}{8}\rho\omega^2 r_2^2=0,$$
$$A+\frac{B}{r_1^2}-\frac{3+\nu}{8}\rho\omega^2 r_1^2=0.$$

The solutions of this system of equations are

$$A=\frac{3+\nu}{8}\rho\omega^2(r_1^2+r_2^2),\qquad B=-\frac{3+\nu}{8}\rho\omega^2 r_1^2 r_2^2.$$

Consequently,

$$\begin{aligned}\sigma_{rr}&=\frac{3+\nu}{8}\rho\omega^2\left(r_1^2+r_2^2-r^2-\frac{r_1^2r_2^2}{r^2}\right),\\ \sigma_{\varphi\varphi}&=\frac{\rho\omega^2}{8}\left[(3+\nu)\left(r_1^2+r_2^2+\frac{r_1^2r_2^2}{r^2}\right)-(1+3\nu)r^2\right].\end{aligned} \tag{6.57}$$

On the basis of these formulas it is easy to verify that the stress σ_{rr} is tensile and attains a maximum value at $r=\sqrt{r_1r_2}$. The stress $\sigma_{\varphi\varphi}$ is also tensile and its maximum value occurs at $r=r_1$. When the hole is very small ($r_1\ll r_2$), the stress $\sigma_{\varphi\varphi}$ changes abruptly at its edge, i.e., stress concentration occurs. It follows from the second formula of (6.57) that

$$\sigma_{\varphi\varphi}^{\max}=\frac{3+\nu}{4}\rho\omega^2 r_2^2.$$

If the disk is solid ($r_1 = 0$), we must take $B = 0$ to obtain a bounded solution; then

$$\sigma_{rr}=\frac{3+\nu}{8}\rho\omega^2(r_2^2-r^2), \qquad \sigma_{\varphi\varphi}=\frac{\rho\omega^2}{8}[(3+\nu)r_2^2-(1+3\nu)r^2].$$

In this case we have, at the centre,

$$\sigma_{rr}=\sigma_{\varphi\varphi}=\frac{3+\nu}{8}\rho\omega^2r_2^2.$$

Thus, in a disk with a very small hole the stress $\sigma_{\varphi\varphi}$ at its edge is twice that at the centre of a solid disk. If the wall of the disk is very thin, it is permissible to put $r_2 \cong r_1$; it follows from the second formula of (6.57) that

$$\sigma_{\varphi\varphi}=\rho\omega^2r_1^2.$$

As a sample problem let us investigate the distribution of stress and displacement in a circular bar under pure bending (Fig. 19). Since the stress tensor is independent of the φ co-ordinate, the stress function is taken in the form of (6.44). We formulate the boundary conditions of the problem as

$$\sigma_{rr}=0 \text{ when } r=R_2,$$
$$\sigma_{rr}=0 \text{ when } r=R_1,$$
$$\int_{R_2}^{R_1}\sigma_{\varphi\varphi}r\,dr=M \text{ when } \varphi=0.$$

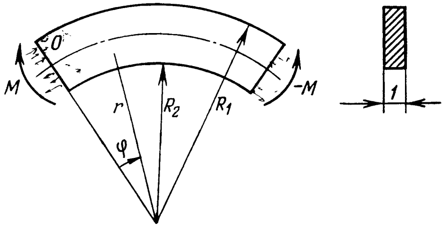

Fig. 19

On the basis of formulas (6.45) these conditions may be put into the form

$$\frac{1}{R_2^2}A+(1+2\ln R_2)B+2C=0,$$
$$\frac{1}{R_1^2}A+(1+2\ln R_1)B+2C=0,$$
$$-A\ln\frac{R_1}{R_2}+B[R_1^2\ln R_1-R_2^2\ln R_2+R_1^2-R_2^2]+(R_1^2-R_2^2)C=M.$$

The solutions of this system of equations are as follows:

$$A=-\frac{4M}{D}R_1^2R_2^2\ln\frac{R_1}{R_2},$$
$$B=-\frac{2M}{D}(R_1^2-R_2^2),$$
$$C=\frac{M}{D}[R_1^2-R_2^2+2(R_1^2\ln R_1-R_2^2\ln R_2)],$$

where

$$D=-(R_1^2-R_2^2)^2+4R_1^2R_2^2\ln^2\frac{R_1}{R_2}.$$

Thus,

$$\sigma_{rr}=-\frac{4M}{D}\left(\frac{R_1^2R_2^2}{r^2}\ln\frac{R_1}{R_2}+R_1^2\ln\frac{r}{R_1}+R_2^2\ln\frac{R_2}{r}\right),$$

$$\sigma_{\varphi\varphi}=-\frac{4M}{D}\left(-\frac{R_1^2R_2^2}{r^2}\ln\frac{R_1}{R_2}+R_1^2\ln\frac{r}{R_1}+R_2^2\ln\frac{R_2}{r}+R_1^2-R_2^2\right).$$

We now determine the displacements u_r and u_φ. For the given problem the formulas of Hooke's law, with (4.50) and (3.31), become

$$\frac{\partial u_r}{\partial r}=\frac{1}{E}\left[(1+\nu)Ar^{-2}+2(1-\nu)B\ln r+(1-3\nu)B+2(1-\nu)C\right],$$

$$\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}=\frac{1}{E}\left[-(1+\nu)Ar^{-2}+2(1-\nu)B\ln r+\right.$$
$$\left.+(3-\nu)B+2(1-\nu)C\right], \tag{6.58}$$

$$\frac{1}{r}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}=0.$$

On integrating successively the first and second equations of this system, there results

$$u_r=\frac{1}{E}\left[-(1+\nu)Ar^{-1}+2(1-\nu)Br\ln r-(1+\nu)Br+\right.$$
$$\left.+2(1-\nu)Cr\right]+f_1'(\varphi),$$

$$u_\varphi=\frac{4B}{E}r\varphi-f_1(\varphi)+f_2(r).$$

Taking into account these relations in the third equation of system (6.58), we obtain

$$\frac{1}{r}f_1''(\varphi)+f_2'(r)+\frac{1}{r}f_1(\varphi)-\frac{1}{r}f_2(r)=0.$$

From this

$$f_1''(\varphi)+f_1(\varphi)=C,\quad rf_2'(r)-f_2(r)=-C.$$

The general solutions of these two equations are, respectively,

$$f_1(\varphi)=P_1\sin\varphi+P_2\cos\varphi+C,\quad f_2(r)=P_3r+C.$$

We thus have

$$u_r=\frac{1}{E}\left[-(1+\nu)Ar^{-1}+2(1-\nu)Br\ln r-(1+\nu)Br+\right.$$
$$\left.+2(1-\nu)Cr\right]+P_1\cos\varphi-P_2\sin\varphi,$$

$$u_\varphi=\frac{4B}{E}r\varphi-P_1\sin\varphi-P_2\cos\varphi+P_3r.$$

To determine the constants P_1, P_2, and P_3 we must take some point, say O, and fix the bar so as to eliminate its motion as a rigid body, i.e., we must put, at this point,

$$u_r = u_\varphi = \frac{\partial u_\varphi}{\partial r} = 0.$$

Then

$$P_1 = -\frac{1}{E}\Big[-(1+\nu)A\frac{2}{R_1+R_2} + 2(1-\nu)B\frac{R_1+R_2}{2}\ln\frac{R_1+R_2}{2} - \\ -(1+\nu)B\frac{R_1+R_2}{2} + 2(1-\nu)C\frac{R_1+R_2}{2}\Big],$$

$$P_2 = P_3 = 0.$$

The displacements become, finally,

$$u_r = \frac{1}{E}[-(1+\nu)Ar^{-1} + 2(1-\nu)Br\ln r - (1+\nu)Br + \\ + 2(1-\nu)Cr] + P_1\cos\varphi,$$

$$u_\varphi = \frac{4B}{E}r\varphi - P_1\sin\varphi.$$

It is seen from the formula for the displacement u_φ that the cross sections remain plane in pure bending.

44. COMPLEX REPRESENTATION OF A BIHARMONIC FUNCTION, OF THE COMPONENTS OF THE DISPLACEMENT VECTOR AND THE STRESS TENSOR

In the preceding section the solution of the equations of plane elasticity was reduced to the boundary problem for the biharmonic equation, which is satisfied by Airy's function. The methods of complex function theory may also be used to advantage in the solution of the equations of plane elasticity. The application of these methods was first given in fundamental investigations of G. V. Kolosov and N. I. Muskhelishvili. The complex representation of the general solution of the equations of plane elasticity was very fruitful for the effective solution of the basic problems in plane elasticity.

In Sec. 32 it was shown that the volume strain for an isotropic homogeneous body in the absence of body forces is a harmonic function; in the case of plane strain we have

$$\frac{\partial^2\theta_1}{\partial x_1^2} + \frac{\partial^2\theta_1}{\partial x_2^2} = 0. \qquad (6.59)$$

The complex representation of solutions of this equation is most easily obtained by writing it in complex form

$$\frac{\partial^2\theta_1}{\partial\bar{z}\,\partial z} = 0, \qquad (6.60)$$

which is directly obtained from (6.59) by introducing new independent complex variables $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ instead of the variables x_1 and x_2, where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

We find from Eq. (6.60) that in a certain region of the plane of the complex variable z the harmonic function may be represented as

$$\theta_1 = \frac{1}{\lambda+\mu}\left[\varphi'(z) + \overline{\varphi'(z)}\right], \tag{6.61}$$

where $\varphi(z)$ is an analytic function of the variable z.

By multiplying the second equation of (6.6) by the imaginary unity i, and adding to the first, with $F_1 = F_2 = 0$, we obtain

$$\mu\Delta(u_1 + iu_2) + (\lambda+\mu)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\theta_1 = 0.$$

Noting that

$$\Delta = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}, \quad \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} = 2\frac{\partial}{\partial \bar{z}},$$

the preceding equation is written in complex form

$$2\mu\frac{\partial^2(u_1 + iu_2)}{\partial z\,\partial \bar{z}} + (\lambda+\mu)\frac{\partial\theta_1}{\partial \bar{z}} = 0.$$

By integrating this equality with respect to the argument $\bar{z}$, we obtain

$$2\mu\cdot\frac{\partial(u_1 + iu_2)}{\partial z} + (\lambda+\mu)\theta_1 = \varphi_1(z), \tag{6.62}$$

where $\varphi_1(z)$ is also an analytic function of the variable z.

Transforming in (6.62) to conjugate expressions, we have

$$2\mu\frac{\partial(u_1 - iu_2)}{\partial \bar{z}} + (\lambda+\mu)\theta_1 = \overline{\varphi_1(z)}. \tag{6.63}$$

By adding together equalities (6.62) and (6.63), and using, along with the relation

$$2\mu\left[\frac{\partial(u_1 + iu_2)}{\partial z} + \frac{\partial(u_1 - iu_2)}{\partial \bar{z}}\right] = 2\mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = 2\mu\theta_1, \tag{6.64}$$

expression (6.61), we find

$$2(\lambda+2\mu)\theta_1 = \frac{2(\lambda+2\mu)}{\lambda+\mu}\left[\varphi'(z) + \overline{\varphi'(z)}\right] = \varphi_1(z) + \overline{\varphi_1(z)}, \tag{6.65}$$

from which

$$\varphi_1(z)=\frac{2(\lambda+2\mu)}{\lambda+\mu}\varphi'(z)+ic; \tag{6.66}$$

here c is a real constant.

Substituting the last formula and also formula (6.61) in equality (6.62), and integrating the result obtained, we have

$$2\mu(u_1+iu_2)=\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}+icz,$$

where $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}=3-4\nu$; $\psi(z)$ is an analytic function of the argument z.

Rejecting the term icz, which gives only a rigid-body displacement, we obtain an important formula for the complex representation of the displacement in a state of plane strain

$$2\mu(u_1+iu_2)=\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}. \tag{6.67}$$

This formula also expresses the displacement in the case of generalized plane stress in a thin plate if $\varkappa$ is replaced by $\varkappa^*$ defined by the relation

$$\varkappa^*=\frac{\lambda^*+3\mu}{\lambda^*+\mu}=\frac{3-\nu}{1+\nu}.$$

Since $\nu<0.5$, it follows that $\varkappa>1$ and $\varkappa^*>1$.

We now proceed to the derivation of formulas for the complex representation of stress components by means of the same pair of analytic functions $\varphi(z)$, $\psi(z)$. For this purpose we write down the formulas of the generalized Hooke's law (6.3) in complex form as follows:

$$\begin{gathered}\sigma_{11}+\sigma_{22}=2(\mu+\lambda)\theta_1,\\ \sigma_{22}-\sigma_{11}+2i\sigma_{12}=2\mu(e_{22}-e_{11}+2ie_{12})=-2\frac{\partial}{\partial z}[2\mu(u_1-iu_2)].\end{gathered} \tag{6.68}$$

Taking into account (6.61) in the first formula of (6.68) and equality (6.67) in the second formula of (6.68), we obtain very important relations giving the complex representation of the components of the stress tensor in a state of plane strain:

$$\begin{gathered}\sigma_{11}+\sigma_{22}=2[\varphi'(z)+\overline{\varphi'(z)}]=4\operatorname{Re}[\varphi'(z)],\\ \sigma_{22}-\sigma_{11}+2i\sigma_{12}=2[\bar{z}\varphi''(z)+\psi'(z)].\end{gathered} \tag{6.69}$$

Formulas (6.67) and (6.69) have found wide application in plane elasticity; they are useful for the reason that the properties of the analytic functions involved in them are well studied.

Let us now express the stress function $\Phi(x_1, x_2)$ in terms of the same analytic functions $\varphi(z)$, $\psi(z)$.

From formulas (6.24) we have

$$\sigma_{11}+\sigma_{22}=\Delta\Phi=4\frac{\partial^2\Phi}{\partial z\,\partial\bar{z}}, \quad \sigma_{22}-\sigma_{11}+2i\sigma_{12}=4\frac{\partial^2\Phi}{\partial z^2}.$$

From these formulas, with (6.69), we obtain

$$2\frac{\partial^2\Phi}{\partial z\,\partial\bar{z}}=\varphi'(z)+\overline{\varphi'(z)}, \quad 2\frac{\partial^2\Phi}{\partial\bar{z}^2}=z\overline{\varphi''(z)}+\overline{\psi'(z)}.$$

By integrating the first equation with respect to z, and the second with respect to $\bar{z}$, we find

$$2\frac{\partial\Phi}{\partial\bar{z}}=\varphi(z)+z\overline{\varphi'(z)}+\overline{g_1(z)},$$

$$2\frac{\partial\Phi}{\partial\bar{z}}=z\overline{\varphi'(z)}+\overline{\psi(z)}+g_2(z).$$

On comparing these equalities we have

$$\varphi(z)-g_2(z)=\overline{\psi(z)}-\overline{g_1(z)}.$$

From this

$$\overline{g_1(z)}=\overline{\psi(z)}+c_1, \quad g_2(z)=\varphi(z)+c_1,$$

and hence

$$2\frac{\partial\Phi}{\partial\bar{z}}=\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}+c_1,$$

from which

$$2\Phi=\bar{z}\varphi(z)+z\overline{\varphi(z)}+\int\overline{\psi(z)}\,d\bar{z}+\chi(z)+c_1\bar{z}. \tag{6.70}$$

Noting that the second derivatives of the stress function (6.24) are real quantities, the function itself must be real, apart from $cz+c_0$. On this account, in expression (6.70) it is necessary to put

$$\overline{\chi(z)}=\int\overline{\psi(z)}\,d\bar{z}+cz+c_0,$$

where c, c_0 are arbitrary complex constants. If we take $c_1=0$, $c=0$, $c_0=0$, without influence on the state of stress, the formula for the complex representation of Airy's stress function becomes, finally,

$$2\Phi=\bar{z}\varphi(z)+z\overline{\varphi(z)}+\chi(z)+\overline{\chi(z)} \tag{6.71}$$

or

$$\Phi=\operatorname{Re}(\bar{z}\varphi(z)+\chi(z)). \tag{6.72}$$

Here the symbol Re indicates that it is necessary to take the real part of the expression which follows it.

Let us derive an expression in complex form for the resultant vector of the forces acting on the side with positive normal on some curve AB (Fig. 20) taken inside the medium in the plane of deformation ox_1x_2. Substituting in relations (6.12) formulas (6.24), which express the components of the stress tensor in terms of the derivatives of Airy's function, and noting that

$$n_1=\frac{dx_2}{dl}, \quad n_2=-\frac{dx_1}{dl},$$

we obtain

$$\begin{aligned} T_{n1}&=\frac{\partial^2\Phi}{\partial x_2^2}\frac{dx_2}{dl}+\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}\frac{dx_1}{dl}=\frac{d}{dl}\left(\frac{\partial\Phi}{\partial x_2}\right),\\ T_{n2}&=-\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}\frac{dx_2}{dl}-\frac{\partial^2\Phi}{\partial x_1^2}\frac{dx_1}{dl}=-\frac{d}{dl}\left(\frac{\partial\Phi}{\partial x_1}\right). \end{aligned} \tag{6.73}$$

By using these formulas, we set up an expression of the form

$$T_{n1}+iT_{n2}=-i\frac{d}{dl}\left(\frac{\partial\Phi}{\partial x_1}+i\frac{\partial\Phi}{\partial x_2}\right).$$

If the components of the resultant vector in question are denoted by (V_1, V_2), from the preceding formula we find

$$V_1+iV_2=\int\limits_{AB}(T_{n1}+iT_{n2})\,dl=$$

$$=-i\left(\frac{\partial\Phi}{\partial x_1}+i\frac{\partial\Phi}{\partial x_2}\right)_A^B,$$

where the symbol $(\ )_A^B$ denotes the increment of the bracketed expression along the curve AB.

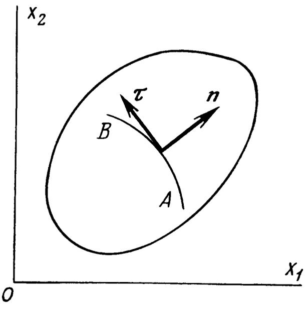

Fig. 20

With (6.71) and the relation $\chi'(z)=\psi(z)$, from the preceding formula we obtain the complex representation of the resultant vector of the forces acting on the curve AB

$$V_1+iV_2=-i\,(\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)})_A^B. \tag{6.74}$$

The resultant moment L_0 of the forces applied to the curve AB on the side with positive normal about the origin is

$$L_0=\int\limits_{AB}(x_1T_{n2}-x_2T_{n1})\,dl.$$

From the last equality, using (6.73) and performing the integration by parts, we obtain

$$L_0 = -\left(x_1 \frac{\partial \Phi}{\partial x_1} + x_2 \frac{\partial \Phi}{\partial x_2}\right)_A^B + (\Phi)_A^B. \tag{6.75}$$

It is obvious that

$$x_1 \frac{\partial \Phi}{\partial x_1} + x_2 \frac{\partial \Phi}{\partial x_2} = \operatorname{Re}\left[z\left(\frac{\partial \Phi}{\partial x_1} - i\frac{\partial \Phi}{\partial x_2}\right)\right].$$

On the other hand, we have, by formula (6.71),

$$\frac{\partial \Phi}{\partial x_1} - i\frac{\partial \Phi}{\partial x_2} = \overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z).$$

This equality together with (6.72) enables us to give formula (6.75) the required complex representation of the resultant moment

$$L_0 = \operatorname{Re}\left(\chi(z) - z\psi(z) - z\bar{z}\varphi'(z)\right)_A^B. \tag{6.76}$$

45. DEGREE OF DETERMINANCY OF THE INTRODUCED FUNCTIONS AND RESTRICTIONS IMPOSED ON THEM

It is easy to show that if the stress tensor σ_{11}, σ_{22}, σ_{12} is given, the function $\varphi'(z)$ is determined except for an additive imaginary constant ci, and the function $\psi'(z)$ is found exactly.

Let $\varphi'(z)$, $\psi'(z)$ be a pair of analytic functions related to the given components σ_{11}, σ_{22}, σ_{12} by formulas (6.69); then

$$\sigma_{11} + \sigma_{22} = 4\operatorname{Re}\varphi'(z), \tag{6.77}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[\bar{z}\varphi''(z) + \psi'(z)\right]. \tag{6.78}$$

Let also $\varphi_1'(z)$, $\psi_1'(z)$ be another pair of functions related to the same σ_{11}, σ_{22}, σ_{12} by the formulas

$$\sigma_{11} + \sigma_{22} = 4\operatorname{Re}\varphi_1'(z), \tag{6.79}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[\bar{z}\varphi_1''(z) + \psi_1'(z)\right]. \tag{6.80}$$

On comparing equalities (6.77) and (6.79), we obtain

$$\varphi_1'(z) = \varphi'(z) + ci.$$

It appears from formulas (6.78) and (6.80) that

$$\psi_1''(z) = \psi'(z).$$

It follows from the last two equalities that

$$\varphi_1(z) = \varphi(z) + ciz + \gamma, \tag{6.81}$$

$$\psi_1(z) = \psi(z) + \gamma', \tag{6.82}$$

where γ, γ' are, in general, complex constants.

The converse statement is also true. If $\varphi(z)$ is replaced by the function $\varphi(z) + ciz + \gamma$, and $\psi(z)$ by the function $\psi(z) + \gamma'$, the state of stress remains unchanged. The validity of this proposition follows from direct substitution of these functions in formulas (6.77) and (6.78).

It is easy to see that if the projections of the displacement vector are given, the constants c, γ, and γ' cannot be prescribed arbitrarily. Let $\varphi(z)$, $\psi(z)$ be a pair of functions related to the given components of the displacement vector by formula (6.67); then

$$2\mu(u_1 + iu_2) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{6.83}$$

By replacing $\varphi(z)$ and $\psi(z)$ according to (6.81) and (6.82), from the preceding formula we obtain

$$2\mu(u_1 + iu_2) = 2\mu(u_1 + iu_2) + (\varkappa + 1)\,ciz + \varkappa\gamma - \overline{\gamma'}.$$

It is seen from this that the projections of the displacement vector remain unchanged if

$$c = 0, \quad \varkappa\gamma - \overline{\gamma'} = 0. \tag{6.84}$$

Thus, in this case only one of the constants γ, γ' may be prescribed arbitrarily. When the stresses are given, it is possible, by a suitable choice of the constants γ, c, γ', to fulfil the conditions

$$\varphi(z_0) = 0, \quad \operatorname{Im}\varphi'(z_0) = 0, \quad \psi(z_0) = 0, \tag{6.85}$$

where z_0 is some fixed point of the region. These conditions completely specify a pair of analytic functions $\varphi(z)$ and $\psi(z)$.

When the projections of the displacement vector are given, it is possible, by choosing one of the constants, γ or γ', to set

$$\varphi(z_0) = 0 \text{ or } \psi(z_0) = 0. \tag{6.86}$$

One of these conditions completely specifies a pair of analytic functions $\varphi(z)$ and $\psi(z)$. If the deformable medium occupies a simply connected region, the functions $\varphi(z)$, $\psi(z)$, $\chi(z)$ are single valued in this region. If a closed curve AA is considered in a simply connected region, where the functions $\varphi(z)$, $\psi(z)$, $\chi(z)$ are single valued, it follows from (6.74) and (6.76) that

$$V_1 + iV_2 = 0, \quad L_0 = 0,$$

i.e., the resultant vector and the resultant moment of the forces applied to this curve are zero. For a multiply connected region, as, for example, in the case of a plate with holes, the functions $\varphi(z)$ and $\psi(z)$ may be multiple valued.

We now turn to the investigation of the nature of multiple-valuedness of these functions, first for the case of a finite multiply connected region and then for an infinite multiply connected region. It is clear that physically the components of the stress tensor must be single valued in the region; the same condition is imposed on the displacement vector. Hence, according to formulas (6.69), along an arbitrary closed curve AA drawn in the multiply connected region occupied by the body, we have

$$(\varphi'(z)+\overline{\varphi'(z)})_A^A=0, \tag{6.87}$$

$$(\bar{z}\varphi''(z)+\varphi'(z))_A^A=0. \tag{6.88}$$

From (6.87) it is apparent that Re $\varphi'(z)$ is a single-valued harmonic function. It is known, however, that the analytic function $\varphi'(z)$ may be multiple valued in a multiply connected region for, on passing round a closed curve situated in the region and enclosing any one of the interior contours, the imaginary part of $\varphi'(z)$, in general, changes by a certain constant amount, and hence the function $\varphi'(z)$ itself receives an increment equal to a purely imaginary constant. We shall see later that in this case no such increment takes place. It appears from the above that the function $\varphi''(z)$ is holomorphic, i.e., a single-valued analytic function. Noting that

$$(\varphi''(z))_A^A=(\overline{\varphi''(z)})_A^A=0, \tag{6.89}$$

from (6.88) we have

$$(\overline{\psi'(z)})_A^A=(\psi'(z))_A^A=0, \tag{6.90}$$

i.e., $\psi'(z)$ is a holomorphic function in a multiply connected region.

By differentiating expression (6.67) with respect to the x_1 co-ordinate, we have

$$2\mu\frac{\partial(u_1+iu_2)}{\partial x_1}=\varkappa\varphi'(z)-\overline{\varphi'(z)}-z\overline{\varphi''(z)}-\overline{\psi'(z)}.$$

Because of the single-valuedness of the quantities $\overline{z\varphi''(z)}+\overline{\psi'(z)}$ [the second formula of (6.69)] and $\frac{\partial}{\partial x_1}(u_1+iu_2)$, from the last equality we have

$$(\varkappa\varphi'(z)-\overline{\varphi'(z)})_A^A=0.$$

Comparison of this equality with (6.87) gives

$$(\varphi'(z))_A^A=(\overline{\varphi'(z)})_A^A=0, \tag{6.91}$$

i.e., $\varphi'(z)$ is also a holomorphic function.

On the basis of (6.91) formulas (6.74) and (6.76) for a closed curve become

$$\begin{aligned} V_1+iV_2&=-i(\varphi(z)+\overline{\psi(z)})_A^A,\\ L_0&=\mathrm{Re}\,(\chi(z)-z\psi(z))_A^A. \end{aligned} \tag{6.92}$$

Let z_k ($k = 1, 2, \ldots, m$) denote the affixes of arbitrarily chosen points inside the corresponding contours L_k having no points in common and enclosed by the outer contour L_0. The function $\varphi(z)$ is written as

$$\varphi(z) = \int_{z_0}^{z} \varphi'(z)\,dz + c, \tag{6.93}$$

where z_0 is an arbitrarily fixed point in the multiply connected region under consideration. The integral

$$\int_{z_0}^{z} \varphi'(z)\,dz$$

is, as a rule, a multiple-valued function, and, on passing round any inner contour L_k, it generally receives an increment $2\pi i A_k$, where A_k is, in general, a complex constant (the factor $2\pi i$ has been introduced for convenience).

It is easy to notice that the function

$$\varphi^*(z) = \int_{z_0}^{z} \varphi'(z)\,dz - \sum_{k=1}^{m} A_k \ln(z - z_k) + c \tag{6.94}$$

is holomorphic in the region under consideration. Indeed, on passing once round the contour L_k the function $A_k \ln(z - z_k)$ receives the same increment $2\pi i A_k$, while the remaining terms in the sum receive no increments, so that the function $\varphi^*(z)$ reverts to its former value.

Taking into account formula (6.94), from equality (6.93) we obtain

$$\varphi(z) = \sum_{k=1}^{m} A_k \ln(z - z_k) + \varphi^*(z), \tag{6.95}$$

where $\varphi^*(z)$ is a holomorphic function. Further, starting from the formula

$$\psi(z) = \int_{z_0}^{z} \psi'(z)\,dz + c',$$

and reasoning in a similar manner, we have

$$\psi(z) = \sum_{k=1}^{m} B_k \ln(z - z_0) + \psi^*(z), \tag{6.96}$$

where B_k are, in general, complex constants and $\psi^*(z)$ is a holomorphic function.

We substitute the expressions for the functions $\varphi(z)$ and $\psi(z)$ in formula (6.67); on passing once round a closed curve L_k' situated in the given region and enclosing only the contour L_k, the displacement vector $u_1 + iu_2$ receives the increment

$$(u_1 + iu_2)_{L'_k} = \frac{\pi}{\mu} i(\varkappa A_k + \bar{B}_k).$$

It is seen from this formula that for displacements to be single valued, the following condition must be fulfilled:

$$\varkappa A_k + \bar{B}_k = 0. \tag{6.97}$$

We now determine the coefficients A_k and $\bar{B}_k$; for this, from the first formula of (6.74) we calculate the resultant vector of the forces applied on the proper side to the same curve L'_k; its magnitude is given by

$$V_{1k} + iV_{2k} = -2\pi(A_k - \bar{B}_k). \tag{6.98}$$

It follows that the resultant vector (V_{1k}, V_{2k}) is independent of the choice of the curve L'_k.

By solving Eqs. (6.97) and (6.98) simultaneously, we obtain

$$A_k = -\frac{V_{1k} + iV_{2k}}{2\pi(1+\varkappa)}, \quad B_k = \frac{\varkappa(V_{1k} - iV_{2k})}{2\pi(1+\varkappa)}. \tag{6.99}$$

Inserting these values of A_k and B_k in formulas (6.95) and (6.96), we have, finally,

$$\varphi(z) = -\frac{1}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (V_{1k} + iV_{2k}) \ln(z - z_k) + \varphi^*(z), \tag{6.100}$$

$$\psi(z) = \frac{z}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (V_{1k} - iV_{2k}) \ln(z - z_k) + \psi^*(z). \tag{6.101}$$

Consider the case of an infinite multiply connected region (for example, the region occupied by an infinite plate weakened by a finite number of curvilinear holes); it can be obtained from the region considered above by taking the outer contour L_0 at infinity. For every point situated outside the circumference L enclosing all boundaries of the holes we have

$$\ln(z - z_k) = \ln z + \ln\left(1 - \frac{z_k}{z}\right).$$

The function $\ln(1 - z_k z^{-1})$ is holomorphic outside the circumference L, including the point at infinity; hence, from formulas (6.100) and (6.101) we find

$$\varphi(z) = -\frac{V_1 + iV_2}{2\pi(1+\varkappa)} \ln z + \varphi^{**}(z), \tag{6.102}$$

$$\psi(z) = \frac{\varkappa(V_1 - iV_2)}{2\pi(1+\varkappa)} \ln z + \psi^{**}(z), \tag{6.103}$$

where (V_1, V_2) is the resultant vector of the forces applied on all contours of the region; $\varphi^{**}(z)$, $\psi^{**}(z)$ are functions holomorphic everywhere outside the circumference L, except possibly at the point at infinity.

Substituting expressions (6.102) and (6.103) in (6.69), and imposing conditions for boundedness of the components of the stress tensor in the whole infinite region under consideration, we arrive at the relations

$$\varphi(z) = -\frac{V_1 + iV_2}{2\pi(1+\varkappa)} \ln z + \Gamma z + \varphi_0(z), \tag{6.104}$$

$$\psi(z) = \frac{\varkappa(V_1 - iV_2)}{2\pi(1+\varkappa)} \ln z + \Gamma' z + \psi_0(z), \tag{6.105}$$

where Γ, Γ' are, in general, complex constants; $\varphi_0(z)$, $\psi_0(z)$ are functions holomorphic outside the circumference L, including the point at infinity, so that the following expansions are valid in its neighbourhood:

$$\begin{aligned} \varphi_0(z) &= a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \\ \psi_0(z) &= b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots. \end{aligned} \tag{6.106}$$

By virtue of formulas (6.85), without changing the state of stress in a medium, we can always assume

$$a_0 = b_0 = 0, \quad \operatorname{Im}\Gamma = 0.$$

The quantity $\operatorname{Im}\Gamma$ has a mechanical meaning. To show this, we proceed as follows. Differentiate relation (6.67) with respect to x_1 and ix_2, and add the resulting expressions. Then

$$\mu\left[\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + i\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right] = \varkappa\varphi'(z) - \overline{\varphi'(z)} =$$
$$= (\varkappa - 1)\operatorname{Re}\varphi'(z) + i(\varkappa + 1)\operatorname{Im}\varphi'(z).$$

From this the value of the rotation ω is determined by the formula

$$\omega = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1+\varkappa}{2\mu}\,\frac{\varphi'(z) - \overline{\varphi'(z)}}{2i}.$$

From this formula, with (6.104), we find, as $z \to \infty$,

$$\omega_\infty = \frac{1+\varkappa}{2\mu}\operatorname{Im}\Gamma.$$

Hence,

$$\operatorname{Im}\Gamma = \frac{2\mu\omega_\infty}{1+\varkappa} \tag{6.106'}$$

Let σ_1, σ_2 be the values of the principal stresses at infinity, and α the angle between the σ_1 direction and the ox_1 axis; we have

$$\sigma_{11}=\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1-\sigma_2}{2}\cos 2\alpha,$$

$$\sigma_{22}=\frac{\sigma_1+\sigma_2}{2}-\frac{\sigma_1-\sigma_2}{2}\cos 2\alpha,$$

$$\sigma_{12}=\frac{\sigma_1-\sigma_2}{2}\sin 2\alpha.$$

On the basis of these formulas

$$\sigma_{11}+\sigma_{22}=\sigma_1+\sigma_2,$$

$$\sigma_{22}-\sigma_{11}+2i\sigma_{12}=-(\sigma_1-\sigma_2)\,e^{-2i\alpha}.$$

By comparing the last expressions with formulas (6.69), and using (6.104), (6.105), and (6.106), we obtain, as $z\to\infty$,

$$\begin{aligned}\operatorname{Re}\Gamma&=\frac{1}{4}(\sigma_1+\sigma_2),\\ \Gamma'&=-\frac{1}{2}(\sigma_1-\sigma_2)e^{-2i\alpha}.\end{aligned}\tag{6.107}$$

Hence, the distribution of the stress tensor in parts of the plane at infinity differs infinitesimally from a uniform distribution.

Substituting formulas (6.104) and (6.105) in (6.67), we have, for large $|z|$,

$$2\mu(u_1+iu_2)=-\frac{\varkappa(V_1+iV_2)}{2\pi(1+\varkappa)}\ln(z\bar{z})+(\varkappa\Gamma-\bar{\Gamma})z-\bar{\Gamma}'\bar{z}+\ldots,\tag{6.108}$$

where the dots indicate terms remaining bounded as $|z|$ increases.

Thus, the displacement at infinity is not bounded; it is bounded if the resultant vector (V_1, V_2) of the forces acting on all contours of the region and the stresses at infinity are zero and if, further, $\operatorname{Im}\Gamma=0$, i.e., the part of the plane at infinity undergoes no rotation.

If the stresses at infinity are zero and the resultant vector of the external forces is not zero, the displacement still increases as $\ln(z\bar{z})=2\ln r$.

46. FUNDAMENTAL BOUNDARY VALUE PROBLEMS AND THEIR REDUCTION TO PROBLEMS OF COMPLEX FUNCTION THEORY

By the fundamental boundary value problems of plane elasticity, as for a three-dimensional body (Sec. 34), we shall understand the following problems:

First fundamental problem. The determination of elastic equilibrium when the external forces applied to the boundary L of a region S are given.

Second fundamental problem. The determination of elastic equilibrium when the displacements of the points of the boundary L are given.

Fundamental mixed problem. The determination of elastic equilibrium when the forces applied on a part of the boundary are given, and the displacements of points on the remainder.

If the region S is infinite, the stresses at infinity must be given in the case of the first fundamental problem, i.e., $\operatorname{Re}\Gamma$ and Γ', and the quantities V_1, V_2, Γ, Γ' in the case of the second fundamental problem and the fundamental mixed problem. Assuming that the solution of the above problems exists, its uniqueness for a finite region can be proved as in the case of the corresponding three-dimensional problems; we shall not consider the proof of the uniqueness theorem for an infinite region; if need be, the reader can find it in the monograph by N. I. Muskhelishvili *Some Basic Problems of the Mathematical Theory of Elasticity.*

It is seen from formulas (6.67), (6.69) that the solution of the plane problem of elasticity reduces to finding a pair of complex functions $\varphi(z)$ and $\psi(z)$, analytic in the given region S, which must satisfy, on its boundary L, certain conditions corresponding to any one of the problems formulated above.

Suppose that the boundary L of the region S is not self-intersecting, is closed, and has a tangent at each point. Besides, we assume that the components of the displacement vector and of the stress tensor are continuous up to the boundary L.

1. For the first fundamental problem, in the case of a finite simply connected region S bounded by a contour L the functions $\varphi(z)$ and $\psi(z)$ must, by (6.74), satisfy the boundary condition

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f_1+if_2+c. \qquad (6.109)$$

Here $t = x_1 + ix_2$ is the affix of a point of L, and x_1 and x_2 are its Cartesian co-ordinates; then

$$f_1+if_2=i\int_0^l (T_{n1}+iT_{n2})\,dl, \qquad (6.110)$$

where T_{n1}, T_{n2} are given values of the projections of the external forces acting on L.

The expression on the left-hand side in (6.109) gives the boundary value of the function $\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}$ when z, remaining inside the region S, tends to a point t of the contour L. This boundary condition exists on account of the above assumption regarding the continuity of the components of the stress tensor up to the contour L. (It should be noted that in formula (6.74) the arc denoted by AB lies entirely within the region S. However, by virtue of the assumption of continuity of the components of the stress tensor up to the contour, we have legitimately applied this formula in the case when the arc AB belongs to the contour L.)

2. For the second fundamental problem, in the case of the same finite simply connected region S the functions $\varphi(z)$ and $\psi(z)$ must, by (6.67), satisfy on the contour L the relation

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu(u_1^* + iu_2^*), \tag{6.111}$$

where u_1^* and u_2^* are given values of the displacement of a point of L.

Here, as above, the left-hand side of equality (6.111) represents the boundary value of the expression

$$\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \text{ as } z \to t.$$

This boundary value exists since

$$\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = 2\mu(u_1 + iu_2),$$

and, according to the condition adopted above, u_1 and u_2 are continuous up to the contour L.

3. For the first fundamental problem, in the case of an infinite region S bounded by a closed contour L the regular functions $\varphi_0(z)$ and $\psi_0(z)$ in it, on the basis of condition (6. 109), with (6.104) and (6.105), must satisfy the boundary relation

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f_1^\bullet + if_2^\bullet + c. \tag{6.112}$$

Here use has been made of the notation

$$f_1^\bullet + if_2^\bullet = f_1 + if_2 + \frac{V_1 + iV_2}{2\pi(1+\varkappa)}(\ln t - \varkappa\ln\bar{t}) + \\ + \frac{V_1 - iV_2}{2\pi(1-\varkappa)}\frac{t}{\bar{t}} - (\Gamma + \bar{\Gamma})t - \bar{\Gamma}'\bar{t}. \tag{6.113}$$

When the point t describes the contour L in the positive sense, the expressions $f_1 + if_2$, $\ln t$, and $\ln\bar{t}$ receive, respectively, increments $i(V_1 + iV_2)$, $-2\pi i$, and $+2\pi i$, so that the increment of the expression $f_1^\circ + if_2^\circ$ is, as can easily be verified, zero. The function $f_1^\circ + if_2^\circ$ is therefore single valued and continuous on L.

4. For the second fundamental problem, in the case of an infinite region S bounded by a contour L the functions $\varphi_0(z)$ and $\psi_0(z)$, on the basis of formula (6.111), with (6.104) and (6.105), must satisfy the boundary condition

$$\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} = 2\mu(u_1^\bullet + iu_2^\bullet), \tag{6.114}$$

where

$$2\mu(u_1^\bullet + iu_2^\bullet) = 2\mu(u_1^* + iu_2^*) + \frac{\varkappa(V_1 + iV_2)}{2\pi(1+\varkappa)}\ln(t\bar{t}) - \\ - \frac{V_1 - iV_2}{2\pi(1+\varkappa)}\frac{t}{\bar{t}} - (\varkappa\Gamma - \bar{\Gamma})t + \bar{\Gamma}'\bar{t}. \tag{6.115}$$

As seen from (6.115), the right-hand side of equality (6.114) represents a single-valued and continuous function on L since such are all the terms in it.

5. In the fundamental mixed problem we have conditions of the form of (6.109) on those parts of the boundary where the projections of the stress vector are given, and conditions of the form of (6.111) on the remainder, where the projections of the displacement vector are given.

As we saw earlier, the condition of continuity of the components of the stress tensor up to the boundary L of the region S leads to the continuity up to the boundary of the expression

$$\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}.$$

and the condition of continuity of the projection of the displacement vector leads to the continuity up to the boundary of the expression

$$\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}.$$

It is obvious that the expression $\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}$ may be continuous up to the boundary without necessarily fulfilling the condition of continuity (up to the boundary L) of the components of the stress tensor. Hence, the latter condition may be replaced by a weaker condition of continuity up to the boundary of the above expression. In the following discussion it is assumed that for the first two fundamental problems the functions $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ are continuously extendible to all points of the boundary L of the region S; this imposes a strong condition on the unknown functions, but considerably simplifies the arguments used in applying efficient methods for solving the fundamental problems.

The condition of continuity of $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ in the case of the first fundamental problem rules out discontinuous external loads, such as concentrated forces; for the mixed problem the functions $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ will not be separately continuous at the points of junction.

47. MAURICE LÉVY'S THEOREM

Consider the first fundamental problem for a finite simply connected region. Since the unknown analytic functions $\varphi(z)$ and $\psi(z)$ are single valued in the given region S and the elastic constants λ and μ do not enter into the boundary condition (6.109), it follows that the solution of this problem given by the functions $\varphi(z)$, $\psi(z)$ is independent of the elastic constants λ and μ, or, in other words, when the external forces are given on the boundary of a finite simply connected region, the state of stress in the body filling it is independent of the elastic properties of the material. For a finite multiply connected region, the solution determined by the functions $\varphi(z)$, $\psi(z)$ depends on the material of the medium. For the solution determined by the functions $\varphi(z)$, $\psi(z)$ to be independent of the elastic constant $\varkappa$, the resultant vectors of the forces applied to each of the contours L_k, as follows from formulas (6.100) and (6.101), must separately be zero. It is in this case that the state of stress is independent of the elastic constants of the body. This result constitutes the theorem of Maurice Lévy, which underlies the method of finding the state of stress at each point of an isotropic homogeneous medium on models of different material. In particular, this theorem makes it possible to replace the determination of the state of stress in homogeneous and isotropic materials by the determination of the state of stress in transparent bodies, optically sensitive to the state of stress set up in them.

The basic methods that furnish the means of solving problems of plane elasticity for a sufficiently wide class of regions are the conformal mapping method and the Cauchy-type integral method. A simultaneous application of these methods proves most effective for simply connected regions.

The investigation of multiply connected regions is much more complicated and will not be considered here.

48. CONFORMAL MAPPING METHOD

Let a finite or an infinite simply connected region in the plane of the variable z, bounded by a simple contour L, be mapped in a one-to-one manner onto the unit circle $|\zeta| < 1$ in the ζ plane by means of the analytic function

$$z = \omega(\zeta), \tag{6.116}$$

assuming that $\omega(0) = 0$ for the finite region and $\omega(0) = \infty$ for the infinite region.

For the infinite region, we consider the case when the displacements u_1, u_2 at infinity are bounded; V_1, V_2, Γ, Γ' in condition (6.108) must then be equal to zero; the components of the stress tensor at infinity are also zero. The unknown functions $\varphi(z)$ and $\psi(z)$ are holomorphic (regular) in the region S, including the point $z = \infty$.

In order to use the conformal mapping (6.116) in the solution of the fundamental problems and, in general, problems of plane elasticity, we transform the boundary conditions (6.109) and (6.111) to the variable ζ.

Introducing the new notation

$$\varphi(z) = \varphi(\omega(\zeta)) = \varphi_1(\zeta), \quad \psi(z) = \psi(\omega(\zeta)) = \psi_1(\zeta), \tag{6.117}$$

we find that the functions $\varphi_1(\zeta)$, $\psi_1(\zeta)$ are holomorphic in the region of the unit circle $|\zeta| < 1$; its boundary is denoted by γ.

We introduce polar co-ordinates (r, θ) in the ζ plane; they may be regarded as the curvilinear co-ordinates of a point (x_1, x_2) of the z plane; since the mapping is conformal, the co-ordinate lines corresponding to r = constant and θ = constant are mutually orthogonal. Take a point (x_1, x_2) in the z plane, and through this point draw co-ordinate lines r = constant and θ = constant (Fig. 21). Denote the projections of the vector $\boldsymbol{a}$, applied at the point z, in the x_1, x_2 co-ordinate system by a_1, a_2, and in the $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ system by a_r, a_θ. It is obvious from Fig. 21 that

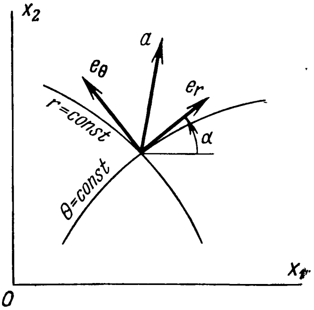

Fig. 21

$$a_r + ia_\theta = (a_1 \cos\alpha + a_2 \sin\alpha) + \\ + i(-a_1 \sin\alpha + a_2 \cos\alpha)$$

or

$$a_r + ia_\theta = e^{-i\alpha}(a_1 + ia_2), \quad (6.118)$$

where α is the angle made by the $\boldsymbol{e}_r$ direction with the x_1 axis and measured from this axis in the positive direction. To calculate $e^{i\alpha}$, we transfer the point z in the $\boldsymbol{e}_r$ direction to the position $z+dz$; the corresponding point $\zeta = re^{i\theta}$ in the ζ plane moves in the radial direction to the position $\zeta + d\zeta$; hence, we have

$$dz = e^{i\alpha}|dz|, \quad d\zeta = e^{i\theta}|d\zeta|,$$

from which, with (6.116),

$$e^{i\alpha} = \frac{dz}{|dz|} = \frac{\omega'(\zeta)\,d\zeta}{|\omega'(\zeta)|\,|d\zeta|} = e^{i\theta}\frac{\omega'(\zeta)}{|\omega'(\zeta)|}.$$

From the last relation we find

$$e^{-i\alpha} = \frac{\bar{\zeta}}{r}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}. \quad (6.119)$$

Substituting (6.119) in (6.118), we obtain

$$a_r + ia_\theta = \frac{\bar{\zeta}}{r}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}(a_1 + ia_2). \quad (6.120)$$

The projections of the displacement vector on the $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ directions are then determined from the equality

$$u_r + iu_\theta = \frac{\bar{\zeta}}{r}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}(u_1 + iu_2). \quad (6.121)$$

On the basis of formulas (1.13), for plane stress and plane strain $(k, l, r, s = 1, 2)$ the following relations hold between the components of the stress tensor σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ in polar co-ordinates and the components of the stress tensor σ_{11}, σ_{22}, σ_{12} in rectangular Cartesian co-ordinates:

$$\begin{aligned}\sigma_{rr}+\sigma_{\theta\theta}&=\sigma_{11}+\sigma_{22},\\ \sigma_{\theta\theta}-\sigma_{rr}+2i\sigma_{r\theta}&=(\sigma_{22}-\sigma_{11}+2i\sigma_{12})\,e^{2i\alpha};\end{aligned}\tag{6.122}$$

the validity of these relations is checked directly. After determining, by (6.119),

$$e^{2i\alpha}=\frac{\zeta^2}{r^2}\,\frac{(\omega'(\zeta))^2}{|\omega'(\zeta)|^2}=\frac{\zeta^2}{r^2}\,\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}},$$

from formulas (6.122), with (6.69), (6.116), and (6.117), we find

$$\begin{aligned}\sigma_{rr}+\sigma_{\theta\theta}&=2\,[\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}],\\ \sigma_{\theta\theta}-\sigma_{rr}+2i\sigma_{r\theta}&=\frac{2\zeta^2}{r^2\,\overline{\omega'(\zeta)}}\,[\overline{\omega(\zeta)}\,\Phi_1'(\zeta)+\omega'(\zeta)\,\Psi_1(\zeta)],\end{aligned}\tag{6.123}$$

where $\Phi_1(\zeta)=\varphi'(z)=\Phi(z)$, $\Psi_1(\zeta)=\psi'(z)=\Psi(z)$.

In this case instead of the boundary conditions (6.109) and (6.111) we have, respectively,

$$\varphi_1(\sigma)+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_1'(\sigma)}+\overline{\psi_1(\sigma)}=f_1^*(\sigma)+if_2^*(\sigma)+c,\tag{6.124}$$

$$\varkappa\varphi_1(\sigma)-\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_1'(\sigma)}-\overline{\psi_1(\sigma)}=2\mu\,[u_1^{**}(\sigma)+iu_2^{**}(\sigma)],\tag{6.125}$$

where

$$\begin{aligned}f_1^*(\sigma)&=f_1(t), & f_2^*(\sigma)&=f_2(t),\\ u_1^{**}(\sigma)&=u_1^*(t), & u_2^{**}(\sigma)&=u_2^*(t)\end{aligned}$$

and σ is the affix of a point of the circumference γ.

It should be noted that since the conformal mapping is a one-to-one mapping, it is necessary that $\omega'(\zeta)\neq 0$. The new unknown analytic functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ corresponding to the old functions $\varphi(z)$ and $\psi(z)$ may be sought in the form of power serie

$$\varphi_1(\zeta)=\sum_{k=1}^{\infty}a_k\zeta^k,\qquad \psi_1(\zeta)=\sum_{k=0}^{\infty}b_k\zeta^k,$$

where the coefficients a_k and b_k are, in general, complex quantities.

The boundary conditions (6.124) or (6.125) enable one to construct an infinite system of linear equations for the determination of these coefficients.

In cases where the mapping function $\omega(\zeta)$ is a polynomial the problem is reduced to a finite system of linear algebraic equations (this result was obtained by N. I. Muskhelishvili). We restrict ourselves to the foregoing general considerations and proceed to the presentation of the theory of the Cauchy-type integral, Harnack's theorem, and the Riemann problem.

49. CAUCHY-TYPE INTEGRAL

As is known, the Dirichlet and Neumann problems for Laplace's equation are solved by means of potentials for a simple and a double layer, and in the solution of boundary value problems for other differential equations use is made of various kinds of generalized potentials. The boundary value problems of the theory of analytic functions of a complex variable, to which problems of plane elasticity are reduced, are solved with the help of the Cauchy-type integral and its various generalizations. Based on this, we present, without proof, some results from the theory of the Cauchy integral, the Cauchy-type integral, and the limiting values of the latter.

1. *Cauchy integral.* Let $f(z)$ be a function, analytic in a simply connected region S bounded by a simple sectionally smooth closed line L, and continuous in $S+L$. The value of the function $f(z)$ at any point $z \in S$ is then determined by the boundary value of this function on the line L as

$$f(z)=\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}. \tag{6.126}$$

Here the integration is carried out along the line L in the positive sense. The integral appearing on the right-hand side of (6.126) is called a Cauchy integral. If the point z is outside L, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}=0 \tag{6.127}$$

because the integrand $f(t)/(t-z)$ is analytic in S and continuous in $S+L$.

In the case of a multiply connected finite region Cauchy's formula is of the form

$$f(z)=\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}, \tag{6.128}$$

where $L=L_0^+ + L_1^- + \ldots + L_m^-$; each of L_ν is a simple sectionally smooth closed line, all L_ν ($\nu = 1, 2, \ldots, m$) being within L_0. If z is a point outside L, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_L \frac{f(t)}{t-z}\,dt=0. \tag{6.129}$$

Let $f(z)$ be a function, analytic in a simply connected infinite region S^- bounded by a simple sectionally smooth closed line L, including the point at infinity, i.e., $f(\infty)=c_0$, and continuous on

$S^- + L$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = -f(z)+f(\infty) \tag{6.130}$$

for points z lying outside L;

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} = f(\infty) \tag{6.131}$$

for points z lying within L. Formula (6.130) is called Cauchy's formula for an infinite region.

2. *Cauchy-type integral.* Let $f(t)$ be a given continuous function on a simple sectionally smooth closed line L; then

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} \tag{6.132}$$

expresses a single-valued analytic function in any simply connected region not containing points of the line L. Integral (6.132) is called a Cauchy-type integral, the function $f(t)$ is called its density function, and $\frac{1}{t-z}$ its kernel; for the derivatives of all orders of the Cauchy-type integral the following formula holds:

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_L \frac{f(t)\,dt}{(t-z)^{n+1}}\,. \tag{6.133}$$

Before proceeding to the study of the behaviour of the Cauchy-type integral on the line of integration, we shall consider the question of classes of functions. Let $f(t)$ be some function, the argument t and the function $f(t)$ being either real or complex. If $f(t)$ is a function of the class of continuous functions, then, by definition, the increments of the argument $|t_2 - t_1|$ and of the function $|f(t_2) - f(t_1)|$ simultaneously tend to zero. The question of the order of smallness of the increment of the function in relation to the increment of the argument is not examined. However, many properties of the function, such as its expansion in series and the rapidity of their convergence, the representation by integrals, etc., are closely related to the order of the modulus of continuity of the function, i.e., $\omega(\delta) = \sup |f(t_2) - f(t_1)|$, where t_1 and t_2 belong to the curve L and $|t_2 - t_1| < \delta$.

We shall consider the most interesting class of functions for which the modulus of continuity is representable as a power-law function of the increment of the argument, i.e.,

$$|f(t_2) - f(t_1)| < A\,|t_2 - t_1|^{\alpha}. \tag{6.134}$$

Here $f(t)$ is a function of the point t on a smooth curve L; t_1, t_2 are any two points of the curve L; A and α are positive numbers. A

is called Hölder's constant, and α Hölder's exponent; $0 < \alpha \leqslant 1$. Condition (6.134) is called a Hölder condition (H condition), and a function $f(t)$ satisfying the H condition is called a function of the class H. Obviously, if $\alpha > 1$, it would appear from condition (6.134) that $f'(t) = 0$ everywhere, and hence $f(t) \equiv$ constant. When $\alpha = 1$, the Hölder condition is identical with the Lipschitz condition. If for sufficiently close t_1 and t_2 the H condition is fulfilled for a certain exponent α_1, it is obviously fulfilled for any exponent $\alpha < \alpha_1$. Thus, to a smaller α corresponds a wider class of functions. The narrowest class is the class of functions satisfying the Lipschitz condition.

3. *The principal value of the Cauchy-type integral.* Let $f(x)$ be a given real function becoming infinite at a certain point c of a finite interval of integration $a \leqslant c \leqslant b$. If we cut out an entirely arbitrary neighbourhood of the point c, the function $f(x)$ is bounded in $a \leqslant x \leqslant c - \varepsilon_1$ and $c + \varepsilon_1 \leqslant x \leqslant b$, and is unbounded in $c - \varepsilon_1 \leqslant x \leqslant c + \varepsilon_2$. The point c is called a singular point.

The limit

$$\lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \left\{ \int_a^{c-\varepsilon_1} f(x)\,dx + \int_{c+\varepsilon_2}^{b} f(x)\,dx \right\}, \tag{6.135}$$

if it exists, is called the improper integral of the function $f(x)$ between the limits a and b. If this limit is finite, it is said that the integral converges, and the function $f(x)$ is termed integrable on the interval $[a, b]$. If, however, the integral is infinite or does not exist at all, it is said that the integral diverges. It is known that the improper integral exists if the order of infinity of the function is less than unity, i.e.,

$$|f(x)| < \frac{M}{|x-c|^{\beta}} \quad (\beta < 1). \tag{6.136}$$

If the function $f(x)$ becomes infinite of order one or higher, the improper integral does not exist.

If a point t_0 of the contour L is substituted for the point z in the curvilinear integral (6.132), we obtain a singular curvilinear integral

$$\int_L \frac{f(t)}{t-t_0}\,dt. \tag{6.137}$$

We represent it as

$$\int_L \frac{f(t)}{t-t_0}\,dt = \int_L \frac{f(t)-f(t_0)}{t-t_0}\,dt + f(t_0) \int_L \frac{dt}{t-t_0}.$$

Since by the Hölder condition

$$|f(t)-f(t_0)| < A\,|t-t_0|^{\alpha}$$

or

$$\left|\frac{f(t)-f(t_0)}{t-t_0}\right|<\frac{A}{|t-t_0|^{\beta}},$$

where $\beta = 1 - \alpha < 1$, it follows that the first integral, on the basis of (6.136), exists as improper.

In the second term the integrand admits the primitive $\ln(t - t_0)$, which is multiple valued. Assume that $\ln(t - t_0)$ is the contour value of the analytic function $\ln(z - t_0)$, single-valued in the plane cut along a curve joining the points t_0 and ∞. We agree, for definiteness, that the cut is made to the right of the line L. Draw a circumference of radius ε from the point t_0 of the line L as a centre, and let t_1 and t_2 be the points of intersection of this circumference and the line L (Fig. 22). Following (6.135), we have

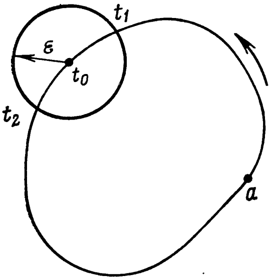

Fig. 22

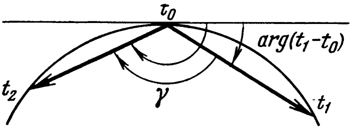

Fig. 23

$$\int_L \frac{dt}{t-t_0} = -\lim_{\varepsilon\to 0}\left[\int_a^{t_1}\frac{dt}{t-t_0}+\int_{t_2}^{a}\frac{dt}{t-t_0}\right]=\lim_{\varepsilon\to 0}\ln\frac{t_1-t_0}{t_2-t_0}$$

or

$$\int_L \frac{dt}{t-t_0}=\lim_{\varepsilon\to 0}\ln\left|\frac{t_1-t_0}{t_2-t_0}\right|+i\lim_{\varepsilon\to 0}[\arg(t_1-t_0)-\arg(t_2-t_0)].$$

Since $|t_2 - t_0| = |t_1 - t_0|$, it follows that $\ln\left|\frac{t_1-t_0}{t_2-t_0}\right| = 0$. The expression in the square brackets is equal to the angle between the vectors t_0t_1, t_0t_2 (Fig. 23), and for the above choice of the cut this angle must be measured to the left of the curve; hence,

$$\lim_{\varepsilon\to 0}\ln\frac{t_1-t_0}{t_2-t_0}=i\pi,$$

and consequently,

$$\int_L \frac{dt}{t-t_0}=i\pi.$$

Thus, the Cauchy principal value of the singular integral (6.137) for the function $f(t)$ satisfying the Hölder condition is

$$\int_L \frac{f(t)}{t-t_0}\,dt=\int_L \frac{f(t)-f(t_0)}{t-t_0}\,dt+i\pi f(t_0). \tag{6.138}$$

4. *The limiting values of the Cauchy-type integral.* Let L be a simple smooth closed line, and $f(t)$ a given function on it satisfying the Hölder condition; the Cauchy-type integral (6.132) then has the limiting values

$$F^-(t_0)=\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0}-\frac{1}{2}f(t_0) \tag{6.139}$$

as $z \to t_0$ from the outside of L, and

$$F^+(t_0)=\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0}+\frac{1}{2}f(t_0) \tag{6.140}$$

as $z \to t_0$ from the inside of L.

Here the singular integral $\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0}$ is understood as a Cauchy principal value and is evaluated by formula (6.138). By adding formulas (6.139) and (6.140) together, we find the value of the Cauchy-type integral at a point lying on the line L

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_0}=\frac{F^+(t_0)+F^-(t_0)}{2}. \tag{6.141}$$

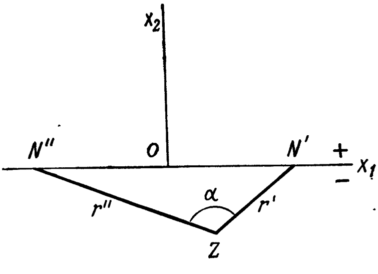

Fig. 24

Consider, now, the Cauchy-type integral for the case when the line of integration is a straight line extending to infinity. Without loss of generality, we take this straight line coincident with the real x_1 axis and denote it by L (Fig. 24). The upper half-plane is denoted by S^+, and the lower half-plane by S^-; the points of the x_1 axis are not included in either S^+ or S^-.

Let $f(t)$ be, in general, a complex function of the real variable t satisfying the H condition for all finite values of t and tending to a definite limit $f(\infty)$ as $t \to \pm\infty$. Moreover, the function $f(t)$ for large values of t satisfies the condition

$$|f(t)-f(\infty)|<\frac{B}{|t|^{\lambda}}, \quad \lambda>0, \quad B>0. \tag{6.142}$$

Assume also that the function $f(t)$ tends to the same finite limit $f(\infty)$ as $t \to +\infty$ and $t \to -\infty$. If $f(\infty) \neq 0$, the Cauchy-type integral

$$\frac{1}{2\pi i} \int_L \frac{f(t)}{t-z}\, dt, \tag{6.143}$$

assuming the point z to lie off the x_1 axis, is divergent. Indeed, we have

$$\int_{N'}^{N''} \frac{f(t)\,dt}{t-z} = \int_{N'}^{N''} \frac{f(t)-f(\infty)}{t-z}\, dt + f(\infty) \int_{N'}^{N''} \frac{dt}{t-z}.$$

In the first integral on the right-hand side the integrand is, by (6.142), of order $|t|^{-1-\lambda}$ for large values of $|t|$; hence, by the well-known criterion for convergence of integrals with infinite limits, the integral in question is convergent. We evaluate the second integral

$$\int_{N'}^{N''} \frac{dt}{t-z} = \ln(N''-z) - \ln(N'-z) = \pm i\alpha + \ln\frac{r''}{r'},$$

where α is the angle included between the straight lines joining the point z to the points N', N'', and r', r'' are the distances of the point z, respectively, to N' and N'' (Fig. 24). In the last equality the sign of the first term is to be taken positive if z lies in the upper half-plane, and negative for the lower half-plane. If N' and N'' independently tend, respectively, to $-\infty$ and $+\infty$, then α tends to π, while $\ln \frac{r''}{r'}$ does not tend to any limit; it follows that the integral in question does not tend to any limit. We now assume that during the whole process $ON' = -ON'' = ON$. Then

$$\lim_{N\to\infty} \ln\frac{r''}{r'} = \ln 1 = 0,$$

and we have

$$\lim_{N\to\infty} \int_{-N}^{N} \frac{f(t)\,dt}{t-z} = \int_{-\infty}^{\infty} \frac{f(t)-f(\infty)}{t-z}\, dt \pm \pi f(\infty). \tag{6.144}$$

If $f(\infty) = 0$ integral (6.143), as the analysis shows, subject to condition (6.142), is convergent.

Thus, the Cauchy singular integral

$$\int_L \frac{f(t)}{t-z}\, dt = \lim_{N\to\infty} \int_{-N}^{N} \frac{f(t)\,dt}{t-z}$$

is called a principal value and is determined by formula (6.144). Under the above conditions imposed on the function $f(t)$, the function $F(z)$ defined by the formula

$$F(z)=\frac{1}{2\pi i}\int\limits_L \frac{f(t)\,dt}{t-z} \tag{6.145}$$

is obviously holomorphic in S^+ and S^-. For this case the Sokhotskii-Plemelj formulas for an infinite straight line are of the form

$$\begin{aligned} F^+(t_0)&=\frac{1}{2}f(t_0)+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{f(t)}{t-t_0}\,dt,\\ F^-(t_0)&=-\frac{1}{2}f(t_0)+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{f(t)}{t-t_0}\,dt. \end{aligned} \tag{6.146}$$

Here $F^+(t_0)$ and $F^-(t_0)$ are the limits of $F(z)$ as $z \to t_0$, respectively, from the upper and lower half-planes.

50. HARNACK'S THEOREM

Let L be a simple closed line. Denote the finite part of the plane bounded by the line L by S^+, and the infinite part of the plane outside this curve by S^-. The line L is not included in either S^+ or S^-. Take a real continuous function of the point on the line L. If

$$\frac{1}{2\pi i}\int\limits_L \frac{f(t)\,dt}{t-z}=0 \text{ for all } z\in S^+,$$

then $f(t) = 0$ everywhere on L. If, however,

$$\frac{1}{2\pi i}\int\limits_L \frac{f(t)\,dt}{t-z}=0 \quad \text{for all } z\in S^-$$

then $f(t) =$ constant on L.

By applying Harnack's theorem to the difference of two real continuous functions $f_1(t)$ and $f_2(t)$ given on L, we have $f_1(t) = f_2(t)$ on L for all $z \in S^+$, and $f_1(t) = f_2(t) + c$ on L for all $z \in S^-$.

Harnack's theorem is formulated in a similar way when L is an infinite straight line.

51. RIEMANN BOUNDARY VALUE PROBLEM

Let L denote a set of a finite number n of simple non-intersecting arcs and closed lines in the plane of the complex variable z. Assume, further, that a definite positive direction is chosen on each of the arcs and lines comprising L. Unclosed arcs are denoted by a_kb_k choosing the notation so as to have the positive direction from a_k to b_k.

A function $F(z)$ is said to be sectionally holomorphic in the whole plane if it is holomorphic in the plane of the complex variable z cut along L, is continuously extendible to all points of L from the left and from the right, with the exception of the ends a_k, b_k, and if the following inequality holds near the ends a_k, b_k:

$$|F(z)| < \frac{B}{|z-c|^{\lambda}} \quad (0 \leqslant \lambda < 1),$$

where c is the affix of any one of the ends a_k, b_k; λ and B are positive constants.

Let $G(t)$ and $f(t)$ be given functions on L satisfying the H condition, with $G(t) \neq 0$ on L. It is required to find a sectionally holomorphic function $F(z)$ whose boundary values on L from the left and from the right, except at the ends a_k, b_k (the concept of the boundary values from the left and from the right is indeterminate), satisfy the condition

$$F^{+}(t) - G(t) F^{-}(t) = f(t), \tag{6.147}$$

where $G(t)$ is known as the coefficient of the Riemann problem, and $f(t)$ its free term. In the case when the function $f(t) = 0$ on L, the problem is said to be homogeneous.

When $G(t) = 1$, we obtain the Riemann problem of the particular kind

$$F^{+}(t) - F^{-}(t) = f(t) \text{ on } L. \tag{6.148}$$

In this case the problem is reduced to the determination of a sectionally holomorphic function $F(z)$ from the given jump $f(t)$ on L. The solution of this problem can be obtained from the Cauchy-type integral

$$F_0(z) = \frac{1}{2\pi i} \int_L \frac{f(t)\,dt}{t-z}. \tag{6.149}$$

Similarly, the function $F_0(z)$ is a sectionally holomorphic function vanishing at infinity and, in addition, it satisfies in the neighbourhood of any end c of the line L the condition

$$|F_0(z)| < \frac{B}{|z-c|^{\lambda}} \tag{6.150}$$

and also

$$F_0^{+}(t) - F_0^{-}(t) = f(t) \tag{6.151}$$

except at the ends a_k, b_k. Consequently, (6.149) is a solution of problem (6.148).

Let us consider the difference $F(z) - F_0(z) = F_*(z)$, where $F(z)$ is the required solution of problem (6.148). On the basis of (6.148) and (6.149) we have, on L,

$$F_*^{+}(t) - F_*^{-}(t) = 0. \tag{6.152}$$

According to the well-known theorem, the values of the function $F_*(z)$ from the left and from the right of L analytically continue each other. Hence, if the function $F_*(z)$ is assigned appropriate values on L and if it is remembered that by virtue of condition (6.150) any end c is a removable singularity, we may consider $F_*(z)$ to be bounded and holomorphic in the whole plane. According to Liouville's theorem, $F_*(z) =$ constant in the whole plane; consequently, $F(z) = F_0(z) + K$ or

$$F(z)=\frac{1}{2\pi i}\int\limits_L \frac{f(t)}{t-z}\,dt+K, \tag{6.153}$$

where K is an arbitrary constant. If it is assumed that the solution $F(z)$ vanishes at the infinitely remote point, we must put $K = 0$.

If the solution of problem (6.148) is to be a sectionally holomorphic function everywhere except at the point at infinity where it may have a pole of order not higher than m, then, by the generalized Liouville theorem,

$$F(z)=\frac{1}{2\pi i}\int\limits_L \frac{f(t)}{t-z}\,dt+C_0+C_1z+\ldots+C_{m-1}z^{m-1}+C_mz^m, \tag{6.154}$$

where $C_0, C_1, \ldots, C_m$ are arbitrary constants.

Of particular interest is the case when $G(t) = g$, where g is a given, in general complex, constant different from unity. Then on L

$$F^+(t) - gF^-(t) = f(t), \tag{6.155}$$

except at the ends. Assuming a pole of order not higher than m at infinity, the general solution of problem (6.155) is of the form

$$F(z)=\frac{X_0(z)}{2\pi i}\int\limits_L \frac{f(t)\,dt}{X_0^+(t)\,(t-z)}+X_0(z)\,P(z), \tag{6.156}$$

where $P(z) = C_0 + C_1z + \ldots + C_{m-1}z^{m-1} + C_mz^m$; $C_0, C_1, \ldots, C_m$ are arbitrary constants; $X_0(z)$ is a particular solution of the homogeneous problem

$$X_0(z)=\prod_{k=1}^{n}(z-a_k)^{-\gamma}(z-b_k)^{\gamma-1}. \tag{6.157}$$

Here

$$\gamma=\frac{\ln|g|}{2\pi i},\quad 0\leqslant \arg g<2\pi. \tag{6.158}$$

The foregoing results are easily extended to the case when the line L is an infinite straight line. In the following discussion these results will be used to solve the fundamental boundary value problems for a half-plane.

52. REDUCTION OF THE FUNDAMENTAL BOUNDARY VALUE PROBLEMS TO FUNCTIONAL EQUATIONS

By using the arbitrariness in regard to the function $\psi(z)$, we can set $\psi(0) = 0$ for a finite region, and $\psi(\infty) = 0$ for an infinite region. Since in the case of a finite region to the point $\zeta = 0$ corresponds the point $z = 0$, and for an infinite region to the same point $\zeta = 0$ corresponds $z = \infty$, we can take $\psi_1(0) = 0$ in both cases.

For an infinite region we assume that the stresses are zero at infinity, the resultant vector of the external forces applied to the boundary is zero, and so is the rotation at infinity. The functions $\varphi(\zeta)$, $\psi(\zeta)$ are then holomorphic inside the circle $|\zeta| < 1$.

We further assume that the functions $\varphi(\zeta)$, $\varphi'(\zeta)$, $\psi(\zeta)$ are continuous up to the circumference γ of the circle under consideration.

(a) We write down the boundary condition (6.124) and its conjugate for the first fundamental problem:

$$\begin{aligned} &\varphi_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_1'(\sigma)} + \overline{\psi_1(\sigma)} = f_1 + if_2, \\ &\overline{\varphi_1(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\varphi_1'(\sigma) + \psi_1(\sigma) = f_1 - if_2. \end{aligned} \tag{6.159}$$

By multiplying both sides of equalities (6.159) by Cauchy's kernel

$$\frac{1}{2\pi i}\,\frac{d\sigma}{\sigma-\zeta},$$

where ζ is a point lying inside the unit circle $|\zeta| < 1$, and integrating them along the circumference γ, we obtain

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\gamma}\frac{\varphi_1(\sigma)}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\varphi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\psi_1(\sigma)}}{\sigma-\zeta}\,d\sigma = A(\zeta), \\ &\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\varphi_1(\sigma)}}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\varphi_1'(\sigma)}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\psi_1(\sigma)}{\sigma-\zeta}\,d\sigma = B(\zeta). \end{aligned} \tag{6.160}$$

Here use has been made of the notation

$$A(\zeta) = \frac{1}{2\pi i}\int_{\gamma}\frac{f_1+if_2}{\sigma-\zeta}\,d\sigma, \qquad B(\zeta) = \frac{1}{2\pi i}\int_{\gamma}\frac{f_1-if_2}{\sigma-\zeta}\,d\sigma.$$

According to Harnack's theorem, relations (6.159) and (6.160) are equivalent. Taking into account that the functions $\varphi_1(\sigma)$ and $\psi_1(\sigma)$ are the boundary values of the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$, regular inside the circle $|\zeta| < 1$, and $\overline{\varphi_1(\sigma)}$, $\overline{\psi_1(\sigma)}$ are the boundary values of the functions regular outside the circle $|\zeta| < 1$ and vanishing at infinity, we ultimately find, using the properties of Cauchy's integral,

$$\varphi_1(\zeta)+\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\varphi_1'(\sigma)}}{\sigma-\zeta}\,d\zeta=A(\zeta), \qquad (6.161)$$

$$\psi_1(\zeta)+\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\varphi_1'(\sigma)}{\sigma-\zeta}\,d\sigma=B(\zeta). \qquad (6.162)$$

The first equation, which is a functional equation, together with the condition $\varphi_1(0)=0$ (in the case of a finite region the quantity Im $\varphi_1'(0)$ may be fixed arbitrarily), completely determines $\varphi_1(\zeta)$, and then the function $\psi_1(\zeta)$ can be found from the second relation.

(b) Proceeding as in the case of the first fundamental problem, for the second fundamental problem we obtain

$$\varkappa\varphi_1(\zeta)-\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\varphi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma=A_1(\zeta),$$

$$\psi_1(\zeta)+\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\varphi_1'(\sigma)}{\sigma-\zeta}\,d\sigma-\varkappa\overline{\varphi}_1(0)=B(\zeta),$$

where

$$A_1(\zeta)=\frac{2\mu}{2\pi i}\int\limits_{\gamma}\frac{u_1+iu_2}{\sigma-\zeta}\,d\sigma,\qquad B(\zeta)=-\frac{2\mu}{2\pi i}\int\limits_{\gamma}\frac{u_1-iu_2}{\sigma-\zeta}\,d\sigma.$$

In an exactly similar way it is possible to obtain a functional equation for the mixed boundary value problem, which is of somewhat more complicated form; we shall not elaborate upon it.

The foregoing functional equations can be reduced by a simple transformation to a Fredholm integral equation of the second kind; we leave detailed discussion at this point.

53. EQUILIBRIUM OF A HOLLOW CIRCULAR CYLINDER

Consider the equilibrium of a hollow circular cylinder subjected to (a) uniformly distributed tangential forces applied on the boundaries; (b) a constant pressure on the boundaries. Both cases come under the first boundary value problem.

In the case (a) the resultant vectors of the forces applied on either of the boundaries $r=r_1$ and $r=r_2$ are separately equal to zero; hence, from formulas (6.100) and (6.101) it follows that the functions $\varphi(z)$ and $\psi(z)$ are holomorphic inside the ring (Fig. 25). The functions $\varphi(z)$ and $\psi(z)$ are determined from the boundary conditions

$$\begin{aligned}&\varphi(t_1)+t_1\overline{\varphi'(t_1)}+\overline{\psi(t_1)}=f(t_1)+c_1 \text{ on circumference } r=r_1,\\&\varphi(t_2)+t_2\overline{\varphi'(t_2)}+\overline{\psi(t_2)}=f(t_2)+c_2 \text{ on circumference } r=r_2,\end{aligned} \qquad (6.163)$$

where

$$f(t_1) = i\int_0^l (T_{11} + iT_{12})\,dl = ir_1T_1\int_0^\alpha (-\sin\alpha + i\cos\alpha)\,d\alpha = iT_1(t_1 - r_1),$$

$$f(t_2) = iT_2(t_2 - r_2).$$

Because of equality of moments, the relation between the tangential forces T_1 and T_2 is of the form

$$T_1r_1^2 = T_2r_2^2.$$

In the circular ring the functions $\varphi(z)$ and $\psi(z)$ are taken in the form

$$\varphi(z) = 0, \quad \psi(z) = \frac{b}{z}. \quad (6.164)$$

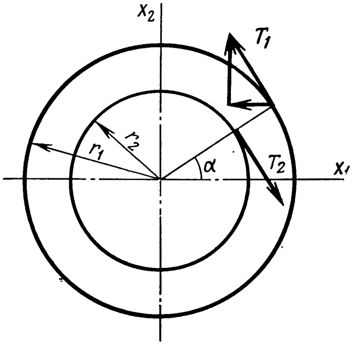

Fig. 25

Inserting (6.164) in the boundary conditions, we find

$$\bar{b}_1 = iT_1r_1^2, \quad c_1 = iT_1r_1,$$

whence

$$b = -iT_1r_1^2.$$

Thus,

$$\varphi(z) = 0, \quad \psi(z) = -iT_1r_1^2\frac{1}{z}.$$

Substituting these functions in formulas (6.123), we obtain

$$\sigma_{rr} + \sigma_{\theta\theta} = 0, \quad \sigma_{rr} - \sigma_{\theta\theta} - 2i\sigma_{r\theta} = -2iT_1r_1^2\frac{1}{z\bar{z}}.$$

From this we find that

$$\sigma_{rr} = \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = \frac{T_1r_1^2}{r^2}.$$

Further, from formula (6.121), with (6.67), we find

$$2\mu(u_r + iu_\theta) = i\frac{T_1r_1^2}{r},$$

whence

$$u_r = 0, \quad u_\theta = \frac{T_1r_1^2}{2\mu}\frac{1}{r}.$$

The problem (b) was solved above by the stress function method, here the same problem is solved by the complex function method. In the problem (b) the resultant vectors and the resultant moments of the forces applied on either of the boundaries $r = r_1$ and $r = r_2$

are separately equal to zero. On the basis of formulas (6.100) and (6.101), for this problem, too, the functions $\varphi(z)$ and $\psi(z)$ are holomorphic inside the ring, and are determined from conditions (6.163); here $f(t_1)$, $f(t_2)$ assume the form

$$f(t_1) = -ip_1r_1\int_0^{\alpha}(\cos\alpha + i\sin\alpha)\,d\alpha = -p_1(t_1 - r_1),\qquad f(t_2) = -p_2(t_2 - r_2). \tag{6.165}$$

The functions $\varphi(z)$ and $\psi(z)$ are taken in the form

$$\varphi(z) = az,\quad \psi(z) = \frac{b}{z}, \tag{6.166}$$

where the coefficients a and b are supposed to be real. For these, substituting (6.164) and (6.166) in the boundary conditions (6.163), we obtain a system of two linear equations

$$2a + b\frac{1}{r_1^2} = -p_1,\quad 2a + b\frac{1}{r_2^2} = -p_2.$$

The roots of this system are

$$a = -\frac{p_2r_2^2 - p_1r_1^2}{2(r_2^2 - r_1^2)},\quad b = -\frac{r_1^2r_2^2(p_1 - p_2)}{r_2^2 - r_1^2},$$

Then

$$\varphi(z) = -\frac{p_2r_2^2 - p_1r_1^2}{2(r_2^2 - r_1^2)}z,\quad \psi(z) = -\frac{r_1^2r_2^2(p_1 - p_2)}{r_2^2 - r_1^2}\frac{1}{z}.$$

Substituting these functions in formulas (6.123) and (6.121), we obtain relations (6.50) and (6.51) with $q = 0$ for the determination of the components of the stress tensor and of the displacement vector.

Consider, now, the following problem.

Let given stresses T_{nr} and $T_{n\theta}$ be applied on the circumference L of a hole of radius R in an infinite plate. The plate is under uniform stress at infinity. Determine the state of stress in the plate.

On the basis of formulas (6.69) and (6.122) we have

$$\sigma_r - i\sigma_{r\theta} = \varphi'(z) + \overline{\varphi'(z)} - [\bar{z}\varphi''(z) + \psi'(z)]e^{2i\alpha}. \tag{6.167}$$

Substituting (6.104) and (6.105) in (6.167), and using (6.106), we have, on the circumference L,

$$-\sum_{k=1}^{\infty}\left[k(k+2)\frac{a_k}{R^{k+1}}\left(\frac{R}{t}\right)^{k+1} + k\frac{\bar{a}_k}{R^{k+1}}\left(\frac{t}{R}\right)^{k+1} - k\frac{b_k}{R^{k+1}}\left(\frac{R}{t}\right)^{k-1}\right] - \frac{V_1 + iV_2}{\pi(1+\varkappa)R}\frac{R}{t} - \frac{V_1 - iV_2}{2\pi R}\frac{t}{R} + 2\operatorname{Re}\Gamma - \Gamma'\left(\frac{t}{R}\right)^2 = T_{nr} - iT_{n\theta}. \tag{6.168}$$

We expand $T_{nr} - iT_{n\theta}$ on the circumference $r = R$ in a Fourier complex series:

$$T_{nr} - iT_{n\theta} = \sum_{k=1}^{\infty}\left[\alpha_k\left(\frac{t}{R}\right)^k + \alpha_{-k}\left(\frac{R}{t}\right)^k\right] + \alpha_0, \qquad (6.169)$$

where

$$\alpha_k = \frac{1}{2\pi}\int_0^{2\pi}(T_{nr} - iT_{n\theta})\left(\frac{t}{R}\right)^k d\theta \quad (k = 0,\ \pm 1,\ \pm 2,\ \ldots). \qquad (6.170)$$

Substituting (6.169) in (6.168), and comparing the coefficients of like powers of $\frac{t}{R}$, $\frac{R}{t}$ on both sides of the resulting relation, we obtain

$$\frac{b_1}{R^2} + 2\operatorname{Re}\Gamma = \alpha_0, \quad 2\frac{b_2}{R^3} - \frac{V_1 + iV_2}{\pi(1+\varkappa)R} = \alpha_{-1}, \quad -\frac{V_1 - iV_2}{2\pi R} = \alpha_1,$$

$$\frac{\bar{a}_1}{R^2} - \Gamma' = \alpha_2, \quad -k\frac{\bar{a}_k}{R^{k+1}} = \alpha_{k+1}, \qquad (6.171)$$

$$-k(k+2)\frac{a_k}{R^{k+1}} + (k+2)\frac{b_{k+2}}{R^{k+3}} = \alpha_{-(k+1)} \quad (k = 2,\ 3,\ \ldots).$$

After determining a_k and b_k from these recursion relations, the components of the stress tensor are found from formulas (6.123). Consider the case when $T_{n\theta} = T_2 =$ constant, $T_{nr} = 0$, and when there is no stress at infinity, i.e., $\Gamma = \Gamma' = 0$; from (6.170) and (6.171), using the fact that $V_1 = V_2 = 0$, we have

$$\alpha_0 = -iT_2, \quad \alpha_k = 0 \quad (k = \mp 1,\ \mp 2,\ \ldots),$$

$$a_k = 0 \quad (k = 1,\ 2,\ \ldots), \quad b_1 = -iT_2R^2, \quad b_k = 0 \quad (k = 2,\ 3,\ \ldots).$$

Consequently,

$$\varphi(z) \equiv 0, \quad \psi(z) = -iT_2\frac{R^2}{z}$$

and

$$\sigma_r = \sigma_\theta = 0, \quad \sigma_{r\theta} = \frac{T_2R^2}{r^2}.$$

The solution obtained coincides with the earlier solution, assuming $r_1 = \infty$, $T_1 = 0$, $r_2 = R$.

For the case $T_{n\theta} = 0$, $T_{nr} = -p$, and $\Gamma = \Gamma' = 0$ we have

$$\alpha_0 = -p, \quad \alpha_k = 0 \quad (k = 1,\ 2,\ \ldots),$$

$$a_k = 0 \quad (k = 1,\ 2,\ 3,\ \ldots), \quad b_1 = -pR^2, \quad b_k = 0 \quad (k = 2,\ 3,\ \ldots).$$

Consequently,

$$\varphi(z) \equiv 0, \quad \psi(z) = -pR^2 \frac{1}{z}$$

and

$$\sigma_{r\theta} = 0, \quad \sigma_r = -pR^2 \frac{1}{r^2}, \quad \sigma_\theta = pR^2 \frac{1}{r^2}.$$

The last formulas coincide with formulas (6.50), assuming $r_2 = \infty$, $p_2 = 0$, $p_1 = p$, and $r_1 = R$.

54. INFINITE PLATE WITH AN ELLIPTIC HOLE

Let us use the conformal mapping method to solve the problem of an unloaded elliptic hole in an infinite plate subjected to equal principal normal stresses p at infinity.

Since the quantity $\operatorname{Im} \Gamma$ does not affect the state of stress, it is taken to be zero; from formulas (6.107) we find

$$\Gamma = \frac{1}{2} p, \quad \Gamma' = 0. \tag{6.172}$$

With the use of the formula

$$z = \omega(\zeta) = A(\zeta^{-1} + m\zeta), \quad A > 0, \quad 0 \leqslant m < 1 \tag{6.173}$$

the outside of the ellipse with centre at the point $z = 0$ and semi-axes $A(1 + m)$, $A(1 - m)$ is conformally mapped onto the unit circle $|\zeta| < 1$.

Remembering that the hole is not loaded, from relations (6.104) and (6.105), with (6.172), we find

$$\varphi(z) = \frac{p}{2} z + \varphi_0(z), \quad \psi(z) = \psi_0(z). \tag{6.174}$$

From this, by virtue of (6.117) and formula (6.173), we have equalities corresponding to (6.174):

$$\varphi_1(\zeta) = \frac{Ap}{2} \zeta^{-1} + \frac{Amp}{2} \zeta + \varphi_0^*(\zeta), \quad \psi_1(\zeta) = \psi_*(\zeta),$$

or

$$\varphi_1(\zeta) = \frac{Ap}{2} \zeta^{-1} + \varphi_*(\zeta), \quad \psi_1(\zeta) = \psi_*(\zeta),$$

$$\varphi_*(\zeta) = \frac{Amp}{2} \zeta + \varphi_0^*(\zeta),$$

where $\varphi_*(\zeta)$ and $\psi_*(\zeta)$ are functions holomorphic inside the circle.

Substituting the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ in (6.124), we see that the functions $\varphi_*(\zeta)$ and $\psi_*(\zeta)$ must satisfy the same equation that is satisfied by the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$, if the right-hand side is replaced by

$$f(\sigma)=f_1^*(\sigma)+if_2^*(\sigma)+\frac{Ap}{2}\left(-\sigma^{-1}+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}\,\overline{\sigma^2}}\right)+C.$$

Taking into account that $f_1^*(\sigma)=f_2^*(\sigma)=0$ for the given problem, and assuming $C=0$, we have, after some manipulation,

$$f(\sigma)=-\frac{Ap}{2}\left[\sigma^{-1}+\frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\right].$$

In the functional equation (6.161) and in relation (6.162), instead of the functions $\varphi_1(\zeta)$, $\psi_1(\zeta)$, $f_1(\sigma)+if_2(\sigma)$ (provided $\varphi_*(0)=0$) we introduce the functions $\varphi_*(\zeta)$, $\psi_*(\zeta)$, $f(\sigma)$, i.e.,

$$\varphi_*(\zeta)+\frac{1}{2\pi i}\int\limits_\gamma\frac{\omega(\sigma)\,\overline{\varphi_*'(\sigma)}}{\overline{\omega'(\sigma)}\,(\sigma-\zeta)}\,d\sigma=\frac{1}{2\pi i}\int\limits_\gamma\frac{f(\sigma)}{\sigma-\zeta}\,d\sigma, \tag{6.175}$$

$$\psi_*(\zeta)=\frac{1}{2\pi i}\int\limits_\gamma\frac{\overline{f(\sigma)}}{\sigma-\zeta}\,d\sigma-\frac{1}{2\pi i}\int\limits_\gamma\frac{\overline{\omega(\sigma)}\,\varphi_*'(\sigma)}{\omega'(\sigma)\,(\sigma-\zeta)}\,d\sigma. \tag{6.176}$$

Substituting (6.173) in Eq. (6.175), we find

$$\varphi_*(\zeta)+\frac{1}{2\pi i}\int\limits_\gamma\frac{(1+m\sigma^2)\,\overline{\varphi_*'(\sigma)}\,d\sigma}{\sigma(m-\sigma^2)(\sigma-\zeta)}=$$

$$=-\frac{pA}{2}\,\frac{1}{2\pi i}\int\limits_\gamma\frac{1}{\sigma-\zeta}\left[\frac{1}{\sigma}+\frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\right]d\sigma. \tag{6.177}$$

The functions

$$\frac{1+m\sigma^2}{\sigma(m-\sigma^2)}\,\overline{\varphi_*'(\sigma)},\quad \frac{1}{\sigma}$$

are the boundary values of the regular functions outside the circumference γ

$$\frac{1+m\zeta^2}{\zeta(m-\zeta^2)}\,\overline{\varphi_*'\left(\frac{1}{\zeta}\right)},\quad \frac{1}{\zeta},$$

which vanish at infinity. The function $\frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\,\frac{1}{\sigma-\zeta}$ has poles $\zeta=\pm\sqrt{m}$, $\zeta=\sigma$ inside γ. Taking into account everything that was said above, and noting that the point ζ lies inside the circle $|\zeta|<1$, from (6.177), using the properties of the Cauchy integral and the residue theorem, we obtain

$$\varphi_*(\zeta)=-\frac{1}{2}\,pAm\zeta. \tag{6.178}$$

Then

$$\varphi_1(\zeta)=\frac{Ap}{2}\left(\frac{1}{\zeta}-m\zeta\right). \tag{6.179}$$

Substituting (6.174), (6.175), and the derivative of function (6.179) in (6.176) we find

$$\psi_*(\zeta) = -\frac{pA}{2}\frac{1}{2\pi i}\int\limits_{\gamma}\left[\sigma+\frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\right]\frac{d\sigma}{\sigma-\zeta}-$$

$$-\frac{pAm}{2}\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\sigma(\sigma^2+m)}{(1-m\sigma^2)(\sigma-\zeta)}d\sigma. \qquad (6.180)$$

Since

$$\frac{\sigma^2+m}{\sigma(1-m\sigma^2)} = \frac{m}{\sigma}+\frac{(1+m^2)\sigma}{1-m\sigma^2}$$

and $m < 1$, the functions σ, $\frac{(1+m^2)\sigma}{1-m\sigma^2}$, $\frac{\sigma(\sigma^2+m)}{1-m\sigma^2}$ are the boundary values of the functions

$$\zeta, \quad \frac{(1+m^2)\zeta}{1-m\zeta^2}, \quad \frac{\zeta(\zeta^2+m)}{1-m\zeta^2},$$

regular inside $|\zeta| < 1$ and m/σ is the boundary value of the function $\frac{m}{\zeta}$, regular outside $|\zeta| < 1$.

Hence, noting that the point ζ lies inside the circle $|\zeta| < 1$, and using Cauchy's formula, from (6.180) we find

$$\psi_1(\zeta) = \psi_*(\zeta) = -Ap(1+m^2)\frac{\zeta}{1-m\zeta^2}. \qquad (6.181)$$

Relations (6.179) and (6.181) can easily be expressed in terms of the basic variable z if for ζ we substitute the inverse function as determined from (6.173):

$$\zeta = \frac{1}{2mA}\left(z-\sqrt{z^2-4mA^2}\right)$$

(the minus sign is taken before the radical as the point z corresponding to $|\zeta| < 1$ is outside the ellipse); the components of the displacement vector and of the stress tensor σ_{11}, σ_{22}, σ_{12} can easily be found from formulas (6.121) and (6.123).

55. SOLUTION OF BOUNDARY VALUE PROBLEMS FOR A HALF-PLANE

The following notation will be used below. Let $\Phi(z) = u(x_1, x_2) + iv(x_1, x_2)$ be a function of the complex variable z defined in some region of the z plane. Then $\overline{\Phi}(z)$ denotes a function assuming values conjugate to $\Phi(z)$ at points $\bar{z}$ conjugate to z, i.e.

$$\overline{\Phi}(z) = \overline{\Phi(\bar{z})} \qquad (6.182)$$

or

$$\overline{\Phi}(z) = u_1(x_1,\ x_2) + iv_1(x_1,\ x_2),$$

where

$$u_1(x_1,\ x_2) = u(x_1,\ -x_2), \qquad v_1(x_1,\ x_2) = -v(x_1,\ -x_2). \tag{6.183}$$

It is readily observed that if $\Phi(z)$ is holomorphic in some region S, then $\overline{\Phi}(z)$ is holomorphic in a region $\overline{S}$ representing a region symmetrical with respect to the region S about the real axis. Indeed, assume that $\Phi(z)$ is holomorphic in S; in the region S we then have

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \qquad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}. \tag{C-R}$$

Taking into account relations (6.183) in the conditions (*C-R*), in the region $\overline{S}$ we have

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial v_1}{\partial x_2}, \qquad \frac{\partial u_1}{\partial x_2} = -\frac{\partial v_1}{\partial x_1}.$$

The last relations show that the functions $u_1(x_1,\ x_2)$ and $v_1(x_1,\ x_2)$ satisfy the conditions (*C-R*) in the region $\overline{S}$.

Let a body occupy the lower half-plane bounded by a straight line which will be taken as the axis of abscissas. Let the lower half-plane $\operatorname{Im}(z) \leqslant 0$ remaining on the right when moving along the ox_1 axis in the positive direction be denoted by S^- and the upper half-plane by S^+.

Let the function $\Phi(z)$ be defined in S^-; the function $\overline{\Phi}(z)$ is then defined in the region S^+. Let there be a boundary value $\Phi^-(t)$ where t is the affix of some point of the ox_1 axis; from formula (6.182) it directly follows that there is a boundary value $\overline{\Phi}^+(t)$ such that

$$\overline{\Phi^-(t)} = \overline{\Phi}^+(t)$$

or

$$\overline{\Phi}^-(t) = \overline{\Phi}^+(t). \tag{6.184}$$

Let complex potentials $\Phi(z)$, $\psi(z)$ be defined in the region S^-, and let there be unloaded parts of the boundary ox_1.

Let us rearrange formulas (6.77), (6.78), and (6.83) in a convenient form to use; for this purpose we construct the analytic continuation of the function $\Phi(z)$ into S^+ through the unloaded parts of the boundary. From formulas (6.77) and (6.78) we have in the region S^-:

$$\sigma_{22} - i\sigma_{12} = \Phi(z) + \overline{\Phi(z)} + \overline{z\Phi'(z)} + \overline{\Psi(z)}, \tag{6.185}$$

where

$$\varphi'(z) = \Phi(z), \qquad \psi'(z) = \Psi(z).$$

Take the function

$$\Phi(z) = -\overline{\Phi}(z) - z\overline{\Phi}'(z) - \overline{\Psi}(z) \tag{6.186}$$

defined by this equality in the region S^+.

It appears from the above that the function $\Phi(z)$ defined by equality (6.186) is holomorphic in the region S^+.

We write $\bar{z}$ for z in (6.186), assuming that z is in S^-, and transform to conjugate values, i.e.,

$$\overline{\Phi}(z) = -\Phi(z) - z\Phi'(z) - \Psi(z):$$

from this

$$\Psi(z) = -\Phi(z) - \overline{\Phi}(z) - z\Phi'(z). \tag{6.187}$$

Formula (6.187) determines the function $\Psi(z)$ in the region S^- through the function $\Phi(z)$ continued into the upper half-plane. On the boundary $\operatorname{Im} z = 0$ expression (6.186) (as $z \to t$ from the region S^+) becomes

$$\Phi^+(t) = -\overline{\Phi}^+(t) - t\overline{\Phi}'^+(t) - \overline{\Psi}^+(t). \tag{6.188}$$

Expression (6.185) on the boundary $\operatorname{Im} z = 0$ is of the form

$$\sigma_{22} - i\sigma_{12}\big|_{x_2=0} = \Phi^-(t) = \overline{\Phi}^-(t) + t\overline{\Phi}'^-(t) + \overline{\Psi}^-(t); \tag{6.189}$$

hence, on parts of the boundary where $\sigma_{22} = \sigma_{12} = 0$ we find

$$\Phi^-(t) = -\overline{\Phi}^-(t) - t\overline{\Phi}'^-(t) - \overline{\Psi}^-(t). \tag{6.190}$$

Comparing (6.188) and (6.190), and using (6.184), by the definition of $\overline{\Phi}^+(t)$ and $\overline{\Psi}^+(t)$, we have

$$\Phi^+(t) = \Phi^-(t). \tag{6.191}$$

Consequently, the function $\Phi(z)$ defined by means of (6.186) in the upper half-plane is the analytic continuation of the function $\Phi(z)$, holomorphic in the lower half-plane, through the unloaded parts of the boundary; in other words, the function $\Phi(z)$ defined by formula (6.186) is a sectionally holomorphic function in the whole plane cut along the loaded parts of the boundary $\operatorname{Im} z = 0$.

It follows from the equality conjugate to expression (6.189) that the function $\Psi(z)$ can be continued through the unloaded parts. Relation (6.185) may be put into a different form. Substituting (6.187) in (6.78) and (6.185), we obtain convenient formulas to use in practice

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[(\bar{z} - z)\Phi'(z) - \Phi(z) - \overline{\Phi}(z)\right], \tag{6.192}$$

$$\sigma_{22} - i\sigma_{12} = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}. \tag{6.193}$$

We now rearrange formula (6.83). For this purpose we continue the function $\varphi(z)$, holomorphic in the region S^-, into the region S^+ so that in this region

$$\varphi'(z) = \Phi(z),$$

where $\Phi(z)$ is given by the right-hand side of (6.186). (As found, in the presence of unloaded parts it analytically continues the unknown function $\Phi(z)$, regular in the lower half-plane, through these parts.) By using formula (6.186), the last relation can be put into the form

$$\varphi'(z) = -[\overline{\Phi}(z) + z\overline{\Phi}'(z) + \overline{\Psi}(z)] = -[z\overline{\varphi}'(z) + \overline{\psi}(z)]';$$

in the region S^+ we then have

$$\varphi(z) = -z\overline{\varphi}'(z) - \overline{\psi}(z) + c_1.$$

It results from this equation that the following relation holds in the region S^-:

$$\psi(z) = -\overline{\varphi}(z) - z\varphi'(z) + \bar{c}_1.$$

With the use of the last relation, formula (6.83) becomes

$$2\mu(u_1 + iu_2) = \varkappa\varphi(z) + \varphi(\bar{z}) - (z - \bar{z})\overline{\varphi'(z)} + c. \qquad (6.194)$$

In the following discussion it is assumed that the function $\Phi(z)$ is continuous from the left and from the right on the contour $\operatorname{Im} z = 0$ except possibly at a finite number of points t_k, and that, in addition,

$$\lim_{x_2 \to 0} x_2\Phi'(z) = 0 \qquad (6.195)$$

for any point of the contour, while the following inequality is valid near the points t_k:

$$|\varphi'(z)| = |\Phi(z)| < \frac{A}{|z - t_k|^{\lambda}} \quad (0 \leqslant \lambda < 1). \qquad (6.196)$$

These conditions ensure that the stress tensor and the displacement vector are continuously extendible to all points of the boundary, with the possible exception of the points t_k.

Suppose that, for large $|z|$, the functions $\Phi(z)$ and $\Psi(z)$ may be represented as

$$\Phi(z) = \frac{\gamma}{z} + o\left(\frac{1}{z}\right), \quad \Psi(z) = \frac{\gamma'}{z} + o\left(\frac{1}{z}\right), \qquad (6.197)$$

where γ, γ' are constants; the symbol $o\left(\frac{1}{z}\right)$ represents a quantity such that $\left|o\left(\frac{1}{z}\right)\right| < \frac{\varepsilon}{|z|}$ (ε depends on $|z|$ and tends to zero as $|z| \to \infty$). To these conditions must be added further conditions: for example, that the following expressions hold for large $|z|$:

$$\begin{aligned} \varphi(z) &= \gamma \ln z + o(1) + \text{constant}, \\ \psi(z) &= \gamma' \ln z + o(1) + \text{constant}, \end{aligned} \tag{6.198}$$

where

$$|o(1)| < \varepsilon \quad (\varepsilon \to 0 \text{ as } |z| \to \infty).$$

With these relations, the components of the stress tensor are zero at infinity.

Let (V_1', V_2') be the resultant vector of the forces applied to a segment AB of the boundary $\operatorname{Im} z = 0$ on the side of the region S^-. Substituting (6.198) in formula (6.74), and letting the end A go to the left and the end B to the right independently of each other, we find

$$V_1' + iV_2' = i(\gamma + \bar{\gamma}') \ln \frac{r''}{r'} + \pi(\bar{\gamma}' - \gamma) + i\varepsilon.$$

Here r' and r'' are, respectively, the distances of the points A and B from the origin and ε is a quantity tending to zero as r' and $r'' \to \infty$. For the resultant vector (V_1', V_2') to remain finite when r' and $r'' \to \infty$ independently of each other, we must assume

$$\gamma + \bar{\gamma}' = 0; \tag{6.199}$$

then

$$V_1' + iV_2' = \pi(\bar{\gamma}' - \gamma).$$

On the other hand, the following relation must be true:

$$V_1' + iV_2' = V_1 + iV_2,$$

where (V_1, V_2) is the resultant vector of the external forces applied to the entire boundary (it will always be finite if the forces are applied over a finite part of the boundary); consequently,

$$V_1 + iV_2 = \pi(\bar{\gamma}' - \gamma). \tag{6.200}$$

From two linear equations, (6.199) and (6.200), we have

$$\gamma = -\frac{V_1 + iV_2}{2\pi}, \quad \bar{\gamma}' = \frac{V_1 - iV_2}{2\pi}. \tag{6.201}$$

1. Solution of the first fundamental problem. In this problem the external forces on the contour L are prescribed as follows:

$$T_{n_2} = \sigma_{22} = -p(t), \quad T_{n_1} = \sigma_{12} = \tau(t),$$

where $p(t)$ and $\tau(t)$ are, respectively, the pressure and the tangential force satisfying the Hölder condition on the contour $\operatorname{Im} z = 0$, including the neighbourhood of the point at infinity. Moreover, $p(\infty) = \tau(\infty) = 0$. According to (6.193) and (6.195), the boundary condition becomes

$$\Phi^+(t)-\Phi^-(t)=p(t)+i\tau(t). \tag{6.202}$$

It is seen from (6.202) that the solution of the first fundamental problem is reduced to the determination of a sectionally holomorphic function from a given jump. The solution of this problem, vanishing at infinity, is, by (6.149),

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{p(t)+i\tau(t)}{t-z}\,dt. \tag{6.203}$$

Knowing the function $\Phi(z)$, the components of the stress tensor (σ_{11}, σ_{22}, σ_{13}) and of the displacement vector (u_1, u_2) can be determined by formulas (6.192), (6.193), and (6.194).

2. Solution of the second fundamental problem. Here the values of the components of the displacement vector on the contour L are prescribed as

$$u_1=g_1(t), \qquad u_2=g_2(t), \tag{6.204}$$

where $g_1(t)$ and $g_2(t)$ are given functions having derivatives which satisfy the Hölder condition, including the point at infinity, and $u_1(\infty)=u_2(\infty)=0$.

On differentiating (6.194) with respect to t the boundary condition (6.204) takes the form

$$\Phi^+(t)+\varkappa\Phi^-(t)=2\mu\,[(g_1'(t)+ig_2'(t)]. \tag{6.205}$$

We introduce a sectionally holomorphic function denoted by $\Omega(z)$ such that

$$\Omega(z)=\begin{cases}\Phi(z) & \text{in } S^-\\ -\dfrac{1}{\varkappa}\Phi(z) & \text{in } S^+.\end{cases} \tag{6.206}$$

The boundary condition (6.205) then becomes

$$\Omega^+(t)-\Omega^-(t)=-\frac{2\mu}{\varkappa}(g_1'(t)+ig_2'(t)).$$

It is seen from this equation that the second fundamental problem is also reduced to the determination of a sectionally holomorphic function from a given jump; its solution is of the form

$$\Omega(z)=-\frac{\mu}{\varkappa\pi i}\int_L \frac{g_1'(t)+ig_2'(t)}{t-z}\,dt \tag{6.207}$$

Thus, from (6.206) we have, finally,

$$\Phi(z)=\begin{cases}\Omega(z) & \text{in } S^-\\ -\varkappa\Omega(z) & \text{in } S^+.\end{cases}$$

3. Solution of the fundamental mixed problem. Let the projections of the displacement vector $g_1(t)$ and $g_2(t)$ be given on a set of a finite number n of segments a_kb_k of the boundary $\operatorname{Im} z = 0$, and the projections of the external force $p(t)$ and $\tau(t)$ on the remaining part of the boundary. The set of segments a_kb_k is denoted by L', and the remaining part of the boundary by L''. Since we already know the solution of the first fundamental problem, it is more convenient to take account of the effect of the given forces on L'' separately; on this consideration we may always assume that the components $p(t) = \tau(t) = 0$ on L''.

Thus, the boundary conditions for this somewhat modified mixed problem take the form

$$u_1 + iu_2 = g(t) + c(t) \quad \text{on} \quad L', \tag{6.208}$$

$$p(t) = 0, \quad \tau(t) = 0 \quad \text{on} \quad L'', \tag{6.209}$$

where $g(t) = g_1(t) + ig_2(t)$ is a function given on L'. The above problem is related to the analysis of punches.

If the quantity $c(t)$ in condition (6.208) is constant on L', it may be taken equal to zero, without loss of generality, for in this case the value of c influences only the rigid-body translation of the whole system. Here it is assumed that, in addition, the resultant vector (V_1, V_2) of the forces applied to L' is given. The fundamental mixed problem in this formulation corresponds to the case of n rigidly connected punches.

When $c(t) = c_k$, where c_k are some constants, it is permissible to fix arbitrarily only one of them on a_kb_k, without loss of generality, and the rest of the constants c_k are to be determined. In this case, in contrast to the preceding one, it is assumed that the resultant vectors (V_{1k}, V_{2k}) of the forces applied to each segment a_kb_k separately are given. This formulation of the problem corresponds to the action of n punches independently undergoing vertical displacements. When $n = 1$, the above problems are identical. On the basis of (6.208) the boundary condition (6.205) on L' for both problems takes the form

$$\Phi^+(t) + \varkappa\Phi^-(t) = 2\mu g'(t). \tag{6.210}$$

By (6.193), with (6.195), the boundary condition (6.209) on L'' is equivalent to the relation $\Phi^+(t) = \Phi^-(t)$, so that the function $\Phi(z)$ is holomorphic in the entire plane cut along L'. Consequently, the solution of the fundamental mixed problem is reduced to a nonhomogeneous Riemann problem.

Suppose that $g'(t)$ satisfies the Hölder condition on L'. The solution of problem (6.210), not vanishing at infinity, can then be represented as

$$\Phi(z) = \frac{\mu X_0(z)}{\pi i} \int\limits_{L'} \frac{g'(t)\,dt}{X_0^+(t)(t-z)} + X_0(z)\,P(z), \tag{6.211}$$

where $X_0(z)$ is a particular solution of the homogeneous problem corresponding to (6.210), holomorphic in the entire plane cut along L'; it is given by

$$X_0(z)=\prod_{k=1}^{n}(z-a_k)^{-\beta}(z-b_k)^{\beta-1},$$

with

$$\beta=\frac{\ln(-\varkappa)}{2\pi i}=\frac{1}{2}-i\frac{\ln\varkappa}{2\pi}.$$

Also, $P(z)$ is some polynomial.

Since the holomorphic function $\Phi(z)$ is to vanish at infinity, the polynomial $P(z)$ must be of degree not higher than $n-1$; hence,

$$P(z)=P_{n-1}(z)=C_0z^{n-1}+C_1z^{n-2}+\ldots+C_{n-1}.$$

The coefficients $C_0, C_1, \ldots, C_{n-1}$ in the polynomial $P_{n-1}(z)$ are determined from additional conditions of the problem. When $c(t)=c_k$ on the segment a_kb_k, for such conditions we take the requirement that the resultant vector (V_{1k}, V_{2k}) of the forces applied to each segment a_kb_k must be equal to prescribed values.

According to (6.202) we have

$$p(t_0)+i\tau(t_0)=\Phi^+(t_0)-\Phi^-(t_0),$$

where t_0 is the affix of a point of L'. Inserting (6.210) in this relation, we have on L'

$$p(t_0)+i\tau(t_0)=\frac{\varkappa+1}{\varkappa}\Phi^+(t_0)-\frac{2\mu}{\varkappa}g'(t_0). \qquad (6.212)$$

By applying the Sokhotskii-Plemelj formula to the right-hand side of (6.211), we find

$$\Phi^+(t_0)=\mu g'(t_0)+\frac{\mu X_0^+(t_0)}{\pi i}\int_{L'}\frac{g'(t)\,dt}{X_0^+(t)(t-t_0)}+X_0^+(t_0)P_{n-1}(t_0). \qquad (6.213)$$

Inserting the value of $\Phi^+(t_0)$ from this in condition (6.212), we obtain

$$p(t_0)+i\tau(t_0)=\frac{\mu(\varkappa-1)}{\varkappa}g'(t_0)+\frac{(\varkappa+1)\mu X_0^+(t_0)}{\varkappa\pi i}\times$$
$$\times\int_{L'}\frac{g'(t)\,dt}{X_0^+(t)(t-t_0)}+\frac{\varkappa+1}{\varkappa}X^+(t_0)P_{n-1}(t_0). \qquad (6.214)$$

Substituting in the obvious relation

$$\int_{a_kb_k}[p(t_0)+i\tau(t_0)]\,dt_0=-V_{2k}+iV_{1k} \quad (k=1, 2, \ldots, n)$$

the value of the integrand from the preceding equality, we arrive at a system of n linear equations for the constants C_{k-1} ($k = 1, 2, \ldots, \ldots, n$); its unique solvability follows from the uniqueness of solution for the original mixed problem.

To solve the problem when $c(t) =$ constant on L' ($c_1 = c_2 = \ldots = c_n$), we calculate the values of $2\mu\,(u_1' + u_2')$ on the unloaded part L'' of the boundary L. Taking into account that in this case the function $\Phi(z)$ is extendible to the unloaded segments of the boundary L'', we obtain, by formulas (6.194) and (6.211),

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$$2\mu\,(u_1'+iu_2') = (\varkappa+1)\,\Phi(t_1) = \\ = \frac{(\varkappa+1)\,\mu X_0^+(t_1)}{\pi i}\int\limits_{L'}\frac{g'(t)\,dt}{X_0^+(t)\,(t-t_1)} + (\varkappa+1)\,X_0^+(t_1)\,P_{n-1}(t_1). \qquad (6.215)$$

Here t_1 is the affix of a point of L''; $u_1' = \frac{\partial u_1}{\partial x_1}$, $u_2' = \frac{\partial u_2}{\partial x_1}$.

On the other hand, on the unloaded segments $b_k a_{k+1}$ we have the obvious relation

$$\int\limits_{b_k}^{a_{k+1}} (u_1'+iu_2')\,dt_1 = g(a_{k+1}) - g(b_k) \quad (k=1,\ 2,\ \ldots,\ n-1).$$

Inserting (6.215) in this relation, we come to a system of $n - 1$ linear equations for C_k ($k = 0, 1, \ldots, n - 1$).

The additional equation is obtained by using a given value of the resultant vector (V_1, V_2) of the forces applied to L'. On the basis of the first formulas of (6.197) and (6.201) we have

$$\lim_{z\to\infty} z\Phi(z) = -\frac{V_1+iV_2}{2\pi};$$

on the other hand, from formula (6.211) we find $\lim\limits_{z\to\infty} z\Phi(z) = C_0$. Consequently,

$$C_0 = -\frac{V_1+iV_2}{2\pi}.$$

Thus, it remains to determine $C_1, C_2, \ldots, C_{n-1}$ from the abovementioned system of $n - 1$ equations whose unique solvability follows from the uniqueness of solution for the original problem.

In the case when $g'(t) = 0$, which corresponds to a straight base of a punch parallel to the boundary $\operatorname{Im} z = 0$, formulas (6.211), (6.213), (6.214), and (6.215), on account of the vanishing integral term in them, are considerably simplified.

56. SOME INFORMATION ON FOURIER INTEGRAL TRANSFORMATION

The Fourier integral transformation is an efficient method for solving elasticity problems when a body is infinite or semi-infinite. We state, without proof, some results pertaining to Fourier integral transformations.

1. The Fourier transform of a certain function $f(x)$ given on the interval $(-\infty, \infty)$ is the integral

$$\bar{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx, \tag{6.216}$$

where ξ is an arbitrary real number.

For the existence of the Fourier transform $\bar{f}(\xi)$ it is sufficient to assume that the function $f(x)$ is absolutely integrable on the interval $(-\infty, \infty)$.

For the function $f(x)$ satisfying, in addition, the Dirichlet conditions* on any finite interval, the following Fourier inversion formula is valid at points of continuity:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\xi)\, e^{-i\xi x}\, d\xi \quad (-\infty<x<+\infty), \tag{6.217}$$

which expresses the function $f(x)$ in terms of its Fourier integral transform. At points of discontinuity the right-hand side of equality (6.217) gives

$$\frac{1}{2}[f(x-0)+f(x+0)].$$

For functions satisfying the Dirichlet conditions on any open interval $0<x<A$ and absolutely integrable on the interval $(0, \infty)$ we have

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_s(\xi)\sin(\xi x)\, d\xi, \quad f_s(\xi)=$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin(\xi x)\, dx, \tag{6.218}$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_c(\xi)\cos(\xi x)\, dx, \quad f_c(\xi)=$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos(\xi x)\, dx.$$

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* If a function $f(x)$ on an interval $(0, A)$ has no more than a finite number of finite discontinuities and a finite number of maxima and minima, this function is said to satisfy the Dirichlet conditions on the given interval.

The function

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi) f(x-\xi)\,d\xi$$

is called the convolution of the functions f and g on the interval $(-\infty, \infty)$.

Theorem. If $\bar{f}(\xi)$ and $\bar{g}(\xi)$ are the Fourier transforms of functions $f(x)$ and $g(x)$, i.e.,

$$\bar{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i\xi x}\,dx,\qquad \bar{g}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{i\xi x}\,dx,$$

then the Fourier transform of the product $\bar{f}(\xi)\,\bar{g}(\xi)$ is the convolution of the functions $f(x)$ and $g(x)$. Indeed, assuming that the inversion of order of integration is permissible, we can write

$$\int_{-\infty}^{\infty} g(\xi) f(x-\xi)\,d\xi=\int_{-\infty}^{\infty} g(\xi)\,d\xi\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(t)\,e^{-it(x-\xi)}\,dt=$$

$$=\int_{-\infty}^{\infty} \bar{f}(t)\,e^{-itx}\,dt\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)\,e^{it\xi}\,d\xi=\int_{-\infty}^{\infty} \bar{f}(t)\,\bar{g}(t)\,e^{-itx}\,dt.$$

Hence,

$$\int_{-\infty}^{\infty} \bar{f}(t)\,\bar{g}(t)\,e^{-itx}\,dt=\int_{-\infty}^{\infty} g(\xi) f(x-\xi)\,d\xi. \tag{6.219}$$

Theorem. The Fourier transform of the function $d^r f/dx^r$ is equal to the Fourier transform of the function $f(x)$ times $(-i\xi)^r$ if $d^k f/dx^k \to 0$ as $x \to \pm\infty$ $(k=0, \ldots, r-1)$, i.e.,

$$\int_{-\infty}^{\infty} \frac{d^r f(x)}{dx^r}\,e^{i\xi x}\,dx=(-i\xi)^r\int_{-\infty}^{\infty} f(x)\,e^{i\xi x}\,dx. \tag{6.220}$$

According to the definition of the Fourier transform,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^r f(x)}{dx^r}\,e^{i\xi x}\,dx=\frac{d^r\bar{f}(\xi)}{d\xi^r}. \tag{6.221}$$

On integrating the left-hand side of (6.221) by parts, we find

$$\left[\frac{1}{\sqrt{2\pi}}\,\frac{d^{r-1}f(x)}{dx^{r-1}}\,e^{i\xi x}\right]_{-\infty}^{\infty}-i\xi\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^{r-1}f(x)}{dx^{r-1}}\,e^{i\xi x}\,dx=\frac{d^r\bar{f}(\xi)}{d\xi^r}.$$

Since $d^{r-1}f(x)/dx^{r-1} \to 0$ as $x \to \pm\infty$, we have

$$-i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^{r-1}f(x)}{dx^{r-1}} e^{i\xi x}\, dx = \frac{d^r \bar{f}(\xi)}{d\xi^r}. \tag{6.222}$$

Integrating, now, the left-hand side of (6.222) by parts, we find

$$(-i\xi)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^{r-2}f(x)}{dx^{r-2}} e^{i\xi x}\, dx = \frac{d^r \bar{f}(\xi)}{d\xi^r}.$$

Repeating this operation, provided that $d^n f(x)/dx^n \to 0$ ($n = 1, 2, \ldots, r-1$) as $x \to \pm\infty$, we obtain, finally

$$(-i\xi)^r \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx = \frac{d^r \bar{f}(\xi)}{d\xi^r}.$$

With (6.221), the last equality takes the form of (6.220).

2. Multiple Fourier transforms. The theory of the Fourier transformation of functions of one variable can be extended to functions of several variables. Suppose, for example, that $f(x_1, x_2)$ is a function of two independent variables, x_1 and x_2; the function f, being considered as a function of x_1, has the Fourier transform

$$\bar{f}(\xi_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1, x_2)\, e^{i\xi_1 x_1}\, dx_1, \tag{6.223}$$

and the function $\bar{f}(\xi_1, x_2)$, being considered as a function of x_2, has the Fourier transform

$$F(\xi_1, \xi_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\xi_1, x_2)\, e^{i\xi_2 x_2}\, dx_2. \tag{6.224}$$

We see from expressions (6.223) and (6.224) that the relation between the functions $f(x_1, x_2)$ and $F(\xi_1, \xi_2)$ is of the form

$$F(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, e^{i(\xi_1 x_1 + \xi_2 x_2)}\, dx_1\, dx_2. \tag{6.225}$$

The function $F(\xi_1, \xi_2)$ is the two-dimensional Fourier transform of the function $f(x_1, x_2)$. On the basis of (6.217) the function $f(x_1, x_2)$ can be expressed in terms of $\bar{f}(\xi_1, x_2)$ by the following formula:

$$f(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\xi_1, x_2)\, e^{-i\xi_1 x_1}\, d\xi_1. \tag{6.226}$$

Similarly, from expression (6.224) we find that

$$\bar{f}(\xi_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi_1, \xi_2)\, e^{-i\xi_2 x_2}\, d\xi_2. \qquad (6.227)$$

From relations (6.226) and (6.227) we derive a formula known as the inversion formula for the two-dimensional Fourier transformation

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi_1, \xi_2)\, e^{-i(\xi_1 x_1 + \xi_2 x_2)}\, d\xi_1\, d\xi_2. \qquad (6.228)$$

The extension of this formula to functions of more variables is obvious.

Let $f = f(x_1, \ldots, x_n)$, then the n-dimensional Fourier transform of this function is

$$F(\xi_1, \ldots, \xi_n) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, e^{i(\xi_1 x_1 + \cdots + \xi_n x_n)}\, dx_1 \ldots dx_n. \qquad (6.229)$$

For this case the inversion formula is of the form

$$f(x_1, \ldots x_n) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(\xi_1, \ldots, \xi_n)\, e^{-i(\xi_1 x_1 + \cdots + \xi_n x_n)}\, d\xi_1 \ldots d\xi_n.$$

We now show that if $F(\xi_1, \xi_2)$ and $G(\xi_1, \xi_2)$ are the Fourier transforms of functions $f(x_1, x_2)$ and $g(x_1, x_2)$, the Fourier transform of the product $F(\xi_1, \xi_2)\, G(\xi_1, \xi_2)$ is the convolution of the functions $f(x_1, x_2)$ and $g(x_1, x_2)$. Indeed, we have, by formula (6.225),

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi_1, \xi_2)\, G(\xi_1, \xi_2)\, e^{-i(\xi_1 x_1 + \xi_2 x_2)}\, d\xi_1\, d\xi_2 =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi_1, \xi_2)\, e^{-i(\xi_1 x_1 + \xi_2 x_2)}\, d\xi_1\, d\xi_2 \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2) \times\right.$$

$$\left. \times\, e^{i(\xi_1 t_1 + \xi_2 t_2)}\, dt_1\, dt_2\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)\, dt_1\, dt_2 \times$$

$$\times \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi_1, \xi_2)\, e^{-i[\xi_1(x_1 - t_1) + \xi_2(x_2 - t_2)]}\, d\xi_1\, d\xi_2\right\}.$$

Taking into account (6.228), we obtain the convolution formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1, t_2)\, g(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi_1, \xi_2)\, G(\xi_1, \xi_2)\, e^{-i(\xi_1 x_1+\xi_2 x_2)}\, d\xi_1\, d\xi_2. \qquad (6.230)$$

As the n-dimensional analogue of formula (6.230) we obtain

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n)\, g(x_1-t_1, \ldots, x_n-t_n)\, dt_1 \ldots dt_n =$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(\xi_1, \ldots, \xi_n)\, G(\xi_1, \ldots, \xi_n)\, e^{-i(\xi_1 x_1+\cdots+\xi_n x_n)}\, d\xi_1 \ldots d\xi_n. \qquad (6.231)$$

57. INFINITE PLANE DEFORMED UNDER BODY FORCES

Let an infinite plane be acted on by given volume forces $\rho F_1(x_1, x_2)$, $\rho F_2(x_1, x_2)$, and let the projections of the displacement vector and the components of the stress tensor tend to zero as $x_1, x_2 \to \infty$. Determine the state of stress for the case of plane strain. By multiplying the equilibrium equations (6.5) and the strain compatibility equation (6.11) by the Fourier kernel $\exp i(\xi_1 x_1 + \xi_2 x_2)$, and integrating them with respect to each of the variables x_1 and x_2 between the limits $-\infty$ and $+\infty$, we come to a system of linear algebraic equations

$$\xi_1\bar{\sigma}_{11} + \xi_2\bar{\sigma}_{12} = -i\rho\bar{F}_1, \quad \xi_1\bar{\sigma}_{12} + \xi_2\bar{\sigma}_{22} = -i\rho\bar{F}_2, \qquad (6.232)$$

$$\bar{\sigma}_{11}[(1-\nu)\xi_2^2 - \nu\xi_1^2] + \bar{\sigma}_{22}[(1-\nu)\xi_1^2 - \nu\xi_2^2] - 2\xi_1\xi_2\bar{\sigma}_{12} = 0,$$

where use has been made of the two-dimensional Fourier transforms of the functions σ_{kr}, F_k $(r, k = 1, 2)$ defined by (6.225):

$$\bar{\sigma}_{kr}(\xi_1, \xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{kr}(x_1, x_2)\exp[i(\xi_1 x_1 + \xi_2 x_2)]\, dx_1\, dx_2,$$

$$\bar{F}_k(\xi_1, \xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_k(x_1, x_2)\exp[i(\xi_1 x_1 + \xi_2 x_2)]\, dx_1\, dx_2.$$

By solving system (6.232), we find

$$\bar{\sigma}_{11}+\bar{\sigma}_{22}=-\frac{\rho}{1-\nu}\,\frac{i\xi_1\bar{F}_1+i\xi_2\bar{F}_2}{\xi_1^2+\xi_2^2}, \tag{6.233}$$

$$\bar{\sigma}_{11}-\bar{\sigma}_{22}=-\frac{\rho(1-2\nu)}{(1-\nu)}\,\frac{\xi_1^2-\xi_2^2}{(\xi_1^2+\xi_2^2)^2}(i\xi_1\bar{F}_1+i\xi_2\bar{F}_2)-4\rho\,\frac{\xi_1\xi_2(i\xi_2\bar{F}_1-i\xi_1\bar{F}_2)}{(\xi_1^2+\xi_2^2)^2}, \tag{6.234}$$

$$\bar{\sigma}_{12}=-\frac{\rho(\xi_1^2-\xi_2^2)}{(\xi_1^2+\xi_2^2)^2}(i\xi_1\bar{F}_2-i\xi_2\bar{F}_1)-\rho\,\frac{1-2\nu}{1-\nu}\,\frac{\xi_1\xi_2(i\xi_1\bar{F}_1+i\xi_2\bar{F}_2)}{(\xi_1^2+\xi_2^2)^2}. \tag{6.235}$$

On the basis of the inversion formula for the two-dimensional Fourier transformation (6.228), from (6.233) we find

$$\sigma_{11}+\sigma_{22}=-\frac{\rho}{2\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi_1\bar{F}_1+i\xi_2\bar{F}_2}{\xi_1^2+\xi_2^2}\exp[-i(\xi_1x_1+\xi_2x_2)]\,d\xi_1\,d\xi_2.$$

Let us show that the function $i\xi_k/(\xi_1^2+\xi_2^2)$ is the Fourier transform of the function $x_k/(x_1^2+x_2^2)$. Indeed

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x_1}{x_1^2+x_2^2}\exp i(\xi_1x_1+\xi_2x_2)\,dx_1\,dx_2=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi_2x_2}\,dx_2\int_{-\infty}^{\infty}\frac{x_1e^{ix_1\xi_1}}{x_1^2+x_2^2}\,dx_1.$$

Here

$$\int_{-\infty}^{\infty}\frac{x_1e^{ix_1\xi_1}}{x_1^2+x_2^2}\,dx_1=$$

$$=\begin{cases} 2\pi i\ \text{res.}\,\dfrac{x_1e^{ix_1\xi_1}}{(x_1+ix_2)(x_1-ix_2)}\Bigg|_{x_1=ix_2}=\begin{cases}\pi i e^{-\xi_1x_2} & \text{when } \begin{matrix}x_2>0\\ \xi_1>0\end{matrix}\\ -\pi i e^{-\xi_1x_2} & \text{when } \begin{matrix}x_2<0\\ \xi_1<0\end{matrix}\end{cases}\\ 2\pi i\ \text{res.}\,\dfrac{x_1e^{ix_1\xi_1}}{(x_1+ix_2)(x_1-ix_2)}\Bigg|_{x_1=-ix_2}=\begin{cases}-\pi i e^{\xi_1x_2} & \text{when } \begin{matrix}x_2>0\\ \xi_1<0\end{matrix}\\ \pi i e^{\xi_1x_2} & \text{when } \begin{matrix}x_2<0\\ \xi_1>0.\end{matrix}\end{cases}\end{cases}$$

Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x_1}{x_1^2+x_2^2}\exp i(\xi_1x_1+\xi_2x_2)\,dx_1\,dx_2=$$

$$=\frac{1}{2}\int_{-\infty}^{0}ie^{x_2(\xi_1+i\xi_2)}\,dx_2+\frac{1}{2}\int_{0}^{\infty}ie^{x_2(-\xi_1+i\xi_2)}\,dx_2=\frac{i\xi_1}{\xi_1^2+\xi_2^2}.$$

Noting that the function $i\xi_k/(\xi_1^2+\xi_2^2)$ is the Fourier transform of the function $x_k/(x_1^2+x_2^2)$, we obtain, by the convolution formula (6.230),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi_k\overline{F}_k(\xi_1,\xi_2)}{\xi_1^2+\xi_2^2}\exp\left[-i(\xi_1x_1+\xi_2x_2)\right]d\xi_1\,d\xi_2=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_k(\alpha_1,\alpha_2)\frac{x_k-\alpha_k}{(x_1-\alpha_1)^2+(x_2-\alpha_2)^2}\,d\alpha_1\,d\alpha_2\qquad(k=1,\ 2).$$

From this formula and calculations similar to those given above [the functions $2i\xi_1\xi_2^2/(\xi_1^2+\xi_2^2)^2$, $i\xi_1(\xi_1^2-\xi_2^2)/2(\xi_1^2+\xi_2^2)^2$ are the Fourier transforms of the functions $x_1(x_1^2-x_2^2)/(x_1^2+x_2^2)^2$ and $x_1x_2^2/(x_1^2+x_2^2)^2$] we obtain

$$\sigma_{11}+\sigma_{22}=$$

$$=-\frac{\rho}{2\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x_1-\alpha_1)F_1(\alpha_1,\alpha_2)+(x_2-\alpha_2)F_2(\alpha_1,\alpha_2)}{(x_1-\alpha_1)^2+(x_2-\alpha_2)^2}\,d\alpha_1\,d\alpha_2,$$

$$\sigma_{11}-\sigma_{22}=-\frac{\rho(1-2\nu)}{\pi(1-\nu)}\times$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x_1-\alpha_1)(x_2-\alpha_2)\left[(x_2-\alpha_2)F_1(\alpha_1,\alpha_2)-(x_1-\alpha_1)F_2(\alpha_1,\alpha_2)\right]}{\left[(x_1-\alpha_1)^2+(x_2-\alpha_2)^2\right]^2}\times$$

$$\times\,d\alpha_1\,d\alpha_2-\frac{\rho}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x_1-\alpha_1)^2-(x_2-\alpha_2)^2}{\left[(x_1-\alpha_1)^2+(x_2-\alpha_2)^2\right]^2}\left[(x_1-\alpha_1)F_1(\alpha_1,\alpha_2)+\right.$$

$$\left.+(x_2-\alpha_2)F_2(\alpha_1,\alpha_2)\right]d\alpha_1\,d\alpha_2,$$

$$\sigma_{12}=-\frac{\rho(1-2\nu)}{4\pi(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x_2-\alpha_2)^2-(x_1-\alpha_1)^2}{\left[(x_1-\alpha_1)^2+(x_2-\alpha_2)^2\right]^2}\left[(x_2-\alpha_2)F_1(\alpha_1,\alpha_2)-\right.$$

$$\left.-(x_1-\alpha_1)F_2(\alpha_1,\alpha_2)\right]d\alpha_1\,d\alpha_2-$$

$$-\frac{\rho}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x_1-\alpha_1)(x_2-\alpha_2)\left[(x_1-\alpha_1)F_1(\alpha_1,\alpha_2)+(x_2-\alpha_2)F_2(\alpha_1,\alpha_2)\right]}{\left[(x_1-\alpha_1)^2+(x_2-\alpha_2)^2\right]^2}\,d\alpha_1\,d\alpha_2.$$

As an example let us consider the problem of an infinite plane acted on by a concentrated force F. The solution of this problem is useful in displaying the nature of stress singularities in the neighbourhood of the point of application of the force. We choose the origin at the point of application of the force, and take the direction of the force coincident with the negative x_1 axis. The mass forces may then be represented as

$$F_1(x_1,x_2)=-\frac{F}{\rho}\,\delta(x_1)\,\delta(x_2),\qquad F_2(x_1,x_2)=0.$$

Here $\delta(x_k)$ is the Dirac function possessing the following properties:

$$\delta(x_k) = 0 \text{ when } x_k \neq 0,$$

$$\int_{-\infty}^{\infty} \delta(x_k)\, dx_k = 1.$$

From the last formulas we obtain, finally,

$$\begin{aligned}
\sigma_{11} &= \frac{Fx_1}{4\pi(1-\nu)(x_1^2+x_2^2)}\left\{(1-2\nu)+\frac{2x_1^2}{x_1^2+x_2^2}\right\},\\
\sigma_{22} &= \frac{Fx_1}{4\pi(1-\nu)(x_1^2+x_2^2)}\left\{(1+2\nu)-\frac{2x_1^2}{x_1^2+x_2^2}\right\},\\
\sigma_{12} &= \frac{Fx_2}{4\pi(1-\nu)(x_1^2+x_2^2)}\left\{(1-2\nu)+\frac{2x_1^2}{x_1^2+x_2^2}\right\}.
\end{aligned} \qquad (6.236)$$

Also, the component of the stress tensor σ_{33} is determined by the well-known formula

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}).$$

It is clear from formulas (6.236) that all components of the stress tensor increase without limit as $x_k \to 0$.

58. SOLUTION OF THE BIHARMONIC EQUATION FOR A WEIGHTLESS HALF-PLANE

To solve the plane problem of elasticity in the absence of body forces, it is necessary, as was established in Sec. 42, to integrate the two-dimensional biharmonic equation (6.26). The solution of this equation will be given for a half-plane bounded by a straight line. Let this half-plane occupy the region $x_1 > 0$ in a rectangular co-ordinate system.

By multiplying (6.26) by $\exp i\xi x_2$, and integrating with respect to the variable x_2 between the limits $-\infty$ and $+\infty$, we obtain

$$\int_{-\infty}^{\infty}\left(\frac{\partial^4\Phi}{\partial x_1^4}+2\frac{\partial^4\Phi}{\partial x_1^2\,\partial x_2^2}+\frac{\partial^4\Phi}{\partial x_2^4}\right)e^{i\xi x_2}\,dx_2 = 0. \qquad (6.237)$$

On the basis of formula (6.220) equality (6.237) can be put into the form

$$\frac{d^4}{dx_1^4}\int_{-\infty}^{\infty}\Phi e^{i\xi x_2}\,dx_2 + 2(-i\xi)^2\frac{d^2}{dx_1^2}\int_{-\infty}^{\infty}\Phi e^{i\xi x_2}\,dx_2 + {}$$

$$+(-i\xi)^4\int_{-\infty}^{\infty}\Phi e^{i\xi x_2}\,dx_2 = 0$$

or

$$\frac{d^4Q}{dx_1^4}-2\xi^2\frac{d^2Q}{dx_1^2}+\xi^4Q=0, \tag{6.238}$$

where

$$Q(x_1, \xi)=\int_{-\infty}^{\infty}\Phi(x_1, x_2)\,e^{i\xi x_2}dx_2 \tag{6.239}$$

The roots of the characteristic equation of the differential equation (6.238) are

$$r_1=r_2=|\xi|, \quad r_3=r_4=-|\xi|.$$

The general solution of (6.238) is then

$$Q=(A+Bx_1)\,e^{-|\xi|x_1}+(C+Dx_1)\,e^{|\xi|x_1}.$$

To obtain a bounded solution, it is necessary to put $C=D=0$, giving

$$Q=(A+Bx_1)\,e^{-|\xi|x_1}. \tag{6.240}$$

The coefficients A and B are determined by the boundary conditions of the problem.

From the Fourier inversion formula (6.217), and (6.239), we find

$$\Phi(x_1, x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Q(x_1, \xi)\,e^{-i\xi x_2}\,d\xi \tag{6.241}$$

Thus, the stress function can be obtained by quadrature.

Now multiply Airy's formulas (6.24) by exp $i\xi x_2$, and integrate with respect to the argument x_2 between the limits $-\infty$ and $+\infty$; taking into account formulas (6.216), (6.220), and (6.239), we find

$$\bar{\sigma}_{11}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2\Phi}{\partial x_2^2}\,e^{i\xi x_2}\,dx_2=-\frac{1}{\sqrt{2\pi}}\,\xi^2Q(x_1, \xi), \tag{6.242}$$

$$\bar{\sigma}_{22}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2\Phi}{\partial x_1^2}\,e^{i\xi x_2}\,dx_2=\frac{1}{\sqrt{2\pi}}\,\frac{d^2Q(x_1, \xi)}{dx_1^2}, \tag{6.243}$$

$$\bar{\sigma}_{12}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}-\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}\,e^{i\xi x_2}\,dx_2=\frac{i\xi}{\sqrt{2\pi}}\,\frac{dQ(x_1, \xi)}{dx_1}. \tag{6.244}$$

From (6.242), (6.243), and (6.244) we obtain, by the Fourier inversion formula,

$$\sigma_{11}(x_1, x_2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{\sigma}_{11}(x_1, \xi)\,e^{-i\xi x_2}\,d\xi=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\xi^2 Q e^{-i\xi x_2}\,d\xi,$$
$$\sigma_{22}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d^2Q}{dx_1^2}\,e^{-i\xi x_2}\,d\xi,\qquad \sigma_{12}=\frac{i}{2\pi}\int_{-\infty}^{\infty}\xi\frac{dQ}{dx_1}\,e^{-i\xi x_2}\,d\xi. \tag{6.245}$$

This method will be used to solve two problems.

1. Half-plane with a distributed force applied to the boundary. Let the x_2 axis be taken along the boundary of the half-plane, with the x_1 axis directed into the half-plane. Suppose that $T_{21} = 0$ and $T_{11} = p\,(x_2)$ are given on the boundary $x_1 = 0$, and there are no body forces acting on the half-plane. Assume that the components of the stress tensor tend to zero as $x_1 \to \infty$. The integration constants A and B in solution (6.240) are determined from the boundary conditions

$$T_{11} = -\sigma_{11}\,(0,\ x_2) = p\,(x_2),\qquad T_{21} = -\sigma_{12}\,(0,\ x_2) = 0.$$

By multiplying these conditions by $\exp i\xi x_2$, and integrating with respect to the variable x_2, we have, for $x_1 = 0$,

$$\bar{\sigma}_{11}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sigma_{11}(0,\ x_2)\,e^{i\xi x_2}\,dx_2=$$
$$=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}p\,(x_2)\,e^{i\xi x_2}\,dx_2=-\bar{p}\,(\xi), \tag{6.246}$$
$$\bar{\sigma}_{12}=0.$$

Substituting (6.240) in (6.242) and (6.244), we find from the boundary conditions (6.246) that

$$A=\frac{\sqrt{2\pi}}{\xi^2}\,\bar{p}\,(\xi),\qquad B=\frac{\sqrt{2\pi}}{\xi^2}\,|\xi|\,\bar{p}\,(\xi). \tag{6.247}$$

Inserting (6.247) in (6.240), and substituting the result obtained in formulas (6.245), we find

$$\sigma_{11}=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{p}\,(\xi)\,[1+|\xi|\,x_1]\,e^{-|\xi|x_1-i\xi x_2}\,d\xi,$$
$$\sigma_{22}=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{p}\,(\xi)\,[1-|\xi|\,x_1]\,e^{-|\xi|x_1-i\xi x_2}\,d\xi, \tag{6.248}$$
$$\sigma_{12}=-\frac{ix_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{p}\,(\xi)\,\xi e^{-|\xi|x_1-i\xi x_2}\,d\xi.$$

If, in particular, the pressure on the boundary $x_1 = 0$ is uniformly distributed over the section $-a \leqslant x_2 \leqslant a$ (Fig. 26), we have, by formula (6.246),

$$\bar{p}(\xi) = \frac{p_0}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\xi x_2}\, dx_2 = \frac{\sqrt{2}}{\sqrt{\pi}} p_0 \frac{\sin(a\xi)}{\xi}. \tag{6.249}$$

Since function (6.249) is even in ξ, formulas (6.248) are rewritten as

$$\sigma_{11} = -\frac{2p_0}{\pi} \int_0^\infty \frac{1+\xi x_1}{\xi} e^{-\xi x_1} \sin(\xi a) \cos(\xi x_2)\, d\xi,$$

$$\sigma_{22} = -\frac{2p_0}{\pi} \int_0^\infty \frac{1-\xi x_1}{\xi} e^{-\xi x_1} \sin(\xi a) \cos(\xi x_2)\, d\xi,$$

$$\sigma_{12} = -\frac{2p_0 x_1}{\pi} \int_0^\infty e^{-\xi x_1} \sin(\xi a) \sin(\xi x_2)\, d\xi.$$

By evaluating, now, these integrals, we find the components of the stress tensor

$$\begin{aligned} \sigma_{11} &= \frac{p_0}{2\pi} [2(\theta_2 - \theta_1) + \sin 2\theta_1 - \sin 2\theta_2], \\ \sigma_{22} &= \frac{p_0}{2\pi} [2(\theta_2 - \theta_1) - \sin 2\theta_1 + \sin 2\theta_2], \\ \sigma_{12} &= \frac{p_0}{2\pi} (\cos 2\theta_2 - \cos 2\theta_1), \end{aligned} \tag{6.250}$$

where, referring to Fig. 26,

$$\theta_1 = \arctan \frac{x_2 - a}{x_1}, \quad \theta_2 = \arctan \frac{x_2 + a}{x_1}.$$

2. Half-plane with a concentrated force applied to the boundary. Consider the stress distribution in a half-plane (Fig. 27) with a concentrated force T applied to the boundary at the origin in the direction of the x_1 axis, when body forces are absent. The solution of this problem can be obtained from solution (6.248) assuming that $p_0 = \frac{\sqrt{\pi}}{\sqrt{2}} T/2a$ and $a \to 0$ in formula (6.249). Then

$$\bar{p}(\xi) = \frac{T}{2} \frac{\sin(a\xi)}{a\xi}\bigg|_{a\to 0} = \frac{T}{2}. \tag{6.251}$$

Inserting (6.251) in formulas (6.248), we find, after evaluating the integrals,

$$\sigma_{11} = -\frac{2T}{\pi} \frac{x_1^3}{r^4}, \quad \sigma_{22} = -\frac{2T}{\pi} \frac{x_1 x_2^2}{r^4}, \quad \sigma_{12} = -\frac{2T}{\pi} \frac{x_1^2 x_2}{r^4}, \tag{6.252}$$

where

$$r=\sqrt{x_1^2+x_2^2}.$$

As $x_1 \to 0$, $x_2 \to 0$, the components of the stress tensor increase without limit. We now calculate the normal and tangential stresses on a plane perpendicular to the radius vector. Let the rotated coordinate axes be taken to coincide with the normal and tangent to the given plane (Fig. 27). From formulas (1.13)

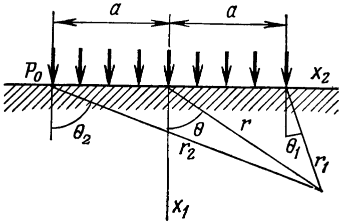


Fig. 26

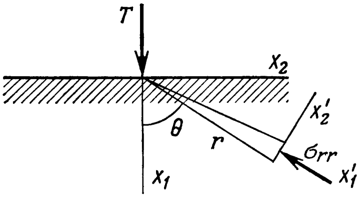


Fig. 27

$$\sigma'_{re}=\sigma_{mk}\alpha_{rm}\alpha_{ek}$$

and from the table given below we have

$$\sigma'_{11}=-\frac{2T}{\pi}\frac{x_1}{r^2}=-\frac{2T}{\pi}\frac{\cos\theta}{r},$$

$$\sigma'_{22}=0, \quad \sigma'_{12}=0.$$

Introducing the notation $\sigma'_{11}=\sigma_{rr}$, $\sigma'_{12}=\sigma_{r\theta}$, $\sigma'_{22}=\sigma_{\theta\theta}$, we obtain the formulas

$$\sigma_{rr}=-\frac{2T}{\pi}\frac{x_1}{r^2}=-\frac{2T}{\pi}\frac{\cos\theta}{r}, \quad \sigma_{r\theta}=\sigma_{\theta\theta}=0.$$

	x_1	x_2	x_3
x_1'	$\alpha_{11}=\cos\theta=\frac{x_1}{r}$	$\alpha_{12}=\sin\theta=\frac{x_2}{r}$	0
x_2'	$\alpha_{21}=-\sin\theta=-\frac{x_2}{r}$	$\alpha_{22}=\cos\theta=\frac{x_1}{r}$	0
x_3'	0	0	1

CHAPTER VII

Torsion and bending of prismatic bodies

59. TORSION OF A PRISMATIC BODY OF ARBITRARY SIMPLY CONNECTED CROSS SECTION

Let the bases of a homogeneous isotropic prismatic body be acted on by forces that reduce to twisting couples. Moreover, body forces are absent and the lateral surface of the body is free from external forces.

Let the ox_3 axis be taken parallel to the generators of the lateral surface, and the ox_1 and ox_2 axes at one of the bases of the bar (Fig. 28).

The problem of the elastic equilibrium of a prismatic body under the above conditions is reduced to that of finding σ_{kr} satisfying, in the region occupied by the body, the differential equations of equilibrium (2.25) in the absence of body forces and the formulas of Hooke's law (4.35), as well as the boundary conditions on the lateral surface and at the bases of the prismatic body.

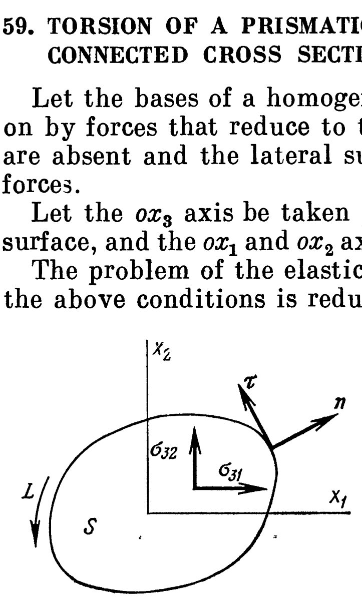


Fig. 28

The problem thus formulated presents great mathematical difficulties. Hence, on the basis of Saint Venant's principle, if the length of the prismatic body is sufficiently great in relation to the dimensions of its bases, we can relax the boundary conditions at the bases so that the resultant vector and the resultant moment of the forces applied to the bases will have prescribed values; the actual distribution of forces at the bases will have little, if any, effect on parts of the body well away from the bases. Such an integral satisfaction of the conditions at the bases permits a sufficient freedom for the choice of solution.

Starting from the above assumptions, Saint Venant solved this problem in terms of displacements by his semi-inverse method. The solution of the formulated problem in terms of displacements is sought by Saint Venant in the form

$$u_1 = -\tau x_2 x_3, \quad u_2 = \tau x_3 x_1, \quad u_3 = \tau \varphi(x_1, x_2), \tag{7.1}$$

where τ is a constant called the degree of twist and $\varphi(x_1, x_2)$ is a function to be determined.

Displacements (7.1) show that the cross sections do not remain plane but warp, and, moreover, all sections warp identically.

From the formulas of Hooke's law (4.35), with formulas (3.26), the components of the stress tensor σ_{kr} corresponding to displacements (7.1) are obtained as

$$\sigma_{31}=\mu\tau\left(\frac{\partial\varphi}{\partial x_1}-x_2\right), \quad \sigma_{32}=\mu\tau\left(\frac{\partial\varphi}{\partial x_2}+x_1\right) \tag{7.2}$$

and

$$\sigma_{11}=\sigma_{22}=\sigma_{33}=\sigma_{12}=0. \tag{7.3}$$

Substituting (7.2) and (7.3) in the differential equations of equilibrium (2.25), when body forces are absent, we see that the first two of them are satisfied identically, and the third equation gives

$$\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}=0. \tag{7.4}$$

The last relation shows that the function $\varphi(x_1, x_2)$, known as Saint Venant's torsion function, must be a harmonic function of the variables x_1 and x_2 in the region S occupied by the cross section of the body. It follows from the third formula of (7.1) that the displacement u_3 must also be a harmonic function.

Noting that the outward normal $\boldsymbol{n}$ to the contour of any cross section is perpendicular to the ox_3 axis, we have $n_3 = 0$. The first two conditions of (2.22), in view of the absence of external forces on the lateral surface and by condition (7.3), are then satisfied identically; the third condition of (2.22) on L, with (7.2), becomes

$$\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)n_1+\left(\frac{\partial\varphi}{\partial x_2}+x_1\right)n_2=0, \tag{7.5}$$

where L denotes the boundary of the region S.

Taking into account that

$$\frac{\partial\varphi}{\partial x_1}n_1+\frac{\partial\varphi}{\partial x_2}n_2=\frac{\partial\varphi}{\partial n},$$

instead of (7.5) on L we obtain

$$\frac{\partial\varphi}{\partial n}=x_2n_1-x_1n_2, \tag{7.6}$$

where $\frac{\partial\varphi}{\partial n}$ is the derivative of φ along the normal n.

The problem of determining the function $\varphi(x_1, x_2)$ is thus the Neumann problem for Laplace's equation. It is easy to show that in our case the condition for the existence of the solution to the Neumann problem is fulfilled. Indeed,

$$\int_L \frac{\partial\varphi}{\partial n}\,dl=\int_L (x_2n_1-x_1n_2)\,dl=\int_L \left(x_2\frac{dx_2}{dl}+x_1\frac{dx_1}{dl}\right)dl=$$

$$=\int_L d\,\frac{1}{2}(x_1^2+x_2^2)=0.$$

If this condition is fulfilled, the solution of the Neumann problem is determined, apart from an arbitrary additive constant. This constant is unimportant since the replacement of the function φ by $\varphi+c$ does not change the state of stress, a result which follows from formulas (7.2), but produces, as the third formula of (7.1) shows, only a rigid-body translation of the body along the ox_3 axis.

The following identity is valid for the harmonic function φ.

$$\frac{\partial}{\partial x_1}\left[x_1\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)\right]+\frac{\partial}{\partial x_2}\left[x_1\left(\frac{\partial\varphi}{\partial x_2}+x_1\right)\right]=\frac{\partial\varphi}{\partial x_1}-x_2; \quad (7.7)$$

on the basis of this identity, with the boundary condition (7.6), we find that the resultant vector of the shearing stresses applied at the cross section is zero. Indeed,

$$V_1=\int_\omega \sigma_{13}\,d\omega=\mu\tau\int_\omega\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)d\omega=$$

$$=\mu\tau\int_\omega\left\{\frac{\partial}{\partial x_1}\left[x_1\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)\right]+\frac{\partial}{\partial x_2}\left[x_1\left(\frac{\partial\varphi}{\partial x_2}+x_1\right)\right]\right\}d\omega.$$

From the Gauss-Ostrogradsky formula and the last equality we obtain

$$V_1=\mu\tau\int_L x_1\left[\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)n_1+\left(\frac{\partial\varphi}{\partial x_2}+x_1\right)n_2\right]dl.$$

In the last formula, using the boundary condition (7.6), we have $V_1=0$; in a similar way it is proved that $V_2=0$. Hence, the shearing stresses applied at the cross section reduce to a couple of moment (Fig. 28)

$$M=\int_\omega (x_1\sigma_{32}-x_2\sigma_{31})\,d\omega. \quad (7.8)$$

Inserting in this formula the values of σ_{32}, σ_{31} from formulas (7.2), we obtain, finally,

$$M=D\tau.$$

In this formula

$$D=\mu\int_\omega\left(x_1^2+x_2^2+x_1\frac{\partial\varphi}{\partial x_2}-x_2\frac{\partial\varphi}{\partial x_1}\right)d\omega. \quad (7.9)$$

From the equilibrium condition we have, at the bases, $M_t = M = D\tau$, from which

$$\tau = \frac{M_t}{D},$$

where M_t is called the twisting moment or torque, and D is the torsional rigidity.

From the Gauss-Ostrogradsky formula, with formula (7.6), we find

$$\int\limits_{\omega} \left(x_1 \frac{\partial\varphi}{\partial x_2} - x_2 \frac{\partial\varphi}{\partial x_1} \right) d\omega = \int\limits_{\omega} \left(\frac{\partial(-x_2\varphi)}{\partial x_1} + \frac{\partial(x_1\varphi)}{\partial x_2} \right) d\omega = -\int\limits_{L} \varphi \frac{\partial\varphi}{\partial n}\, dl.$$

On the other hand, by Green's first formula,

$$\int\limits_{L} \varphi \frac{\partial\varphi}{\partial n}\, dl = \int\limits_{\omega} \left[\left(\frac{\partial\varphi}{\partial x_1} \right)^2 + \left(\frac{\partial\varphi}{\partial x_2} \right)^2 \right] d\omega.$$

Consequently,

$$\int\limits_{\omega} \left[\left(\frac{\partial\varphi}{\partial x_1} \right)^2 + \left(\frac{\partial\varphi}{\partial x_2} \right)^2 + x_1 \frac{\partial\varphi}{\partial x_2} - x_2 \frac{\partial\varphi}{\partial x_1} \right] d\omega = 0.$$

By multiplying both sides of the last relation by μ, and adding to (7.9), we obtain

$$D = \mu \int\limits_{\omega} \left[\left(\frac{\partial\varphi}{\partial x_1} - x_2 \right)^2 + \left(\frac{\partial\varphi}{\partial x_2} + x_1 \right)^2 \right] d\omega.$$

It follows that D is always positive.

We introduce a harmonic function $\psi(x_1, x_2)$ conjugate to the function $\varphi(x_1, x_2)$; by the Cauchy-Riemann conditions we then have

$$\frac{\partial\varphi}{\partial x_1} = \frac{\partial\psi}{\partial x_2}; \quad \frac{\partial\varphi}{\partial x_2} = -\frac{\partial\psi}{\partial x_1}. \tag{7.10}$$

The boundary condition that is satisfied by the function $\psi(x_1, x_2)$ is obtained from (7.6) by inserting conditions (7.10) in (7.6), and using (6.27). The result is

$$\frac{\partial\psi}{\partial x_2} \frac{dx_2}{dl} + \frac{\partial\psi}{\partial x_1} \frac{dx_1}{dl} - \left(x_2 \frac{dx_2}{dl} + x_1 \frac{dx_1}{dl} \right) = 0.$$

By integrating both sides of this equality along the contour of the cross section, we have

$$\psi|_L = \frac{x_1^2 + x_2^2}{2} + C. \tag{7.11}$$

By (7.2), with conditions (7.10), the components of the stress tensor are obtained as

$$\sigma_{31} = \mu\tau \left(\frac{\partial\psi}{\partial x_2} - x_2 \right), \quad \sigma_{32} = -\mu\tau \left(\frac{\partial\psi}{\partial x_1} - x_1 \right). \tag{7.12}$$

It is well seen from these formulas that the solution of the problem will not change if a constant is added to the function $\psi(x_1, x_2)$. Consequently, the determination of the function is reduced to the Dirichlet problem for Laplace's equation.

Instead of the function $\psi(x_1, x_2)$ another function, $\Phi(x_1, x_2)$, is often introduced, called the stress function in torsion or Prandtl's stress function. This function is defined by the formula

$$\Phi(x_1, x_2) = \psi(x_1, x_2) - \frac{1}{2}(x_1^2 + x_2^2). \tag{7.13}$$

In this case, from (7.12) we have

$$\sigma_{31} = \mu\tau \frac{\partial \Phi}{\partial x_2}, \quad \sigma_{32} = -\mu\tau \frac{\partial \Phi}{\partial x_1}. \tag{7.14}$$

From (7.13), noting that

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0,$$

we obtain

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} = -2 \tag{7.15}$$

By (7.11), the boundary condition for the function Φ becomes

$$\Phi(x_1, x_2)\,|_L = C. \tag{7.16}$$

Thus, the problem of determining $\Phi(x_1, x_2)$ is the Dirichlet problem for Poisson's equation (7.15) subject to the boundary condition (7.16). From formula (7.8), with (7.14), the twisting moment is determined as

$$M_t = -\mu\tau \int_\omega \left(\frac{\partial \Phi}{\partial x_1} x_1 + \frac{\partial \Phi}{\partial x_2} x_2 \right) d\omega. \tag{7.17}$$

This formula shows that the magnitude of the moment M_t does not change if any constant is added to the function Φ. By writing (7.17) as

$$M_t = -\mu\tau \int_\omega \left[\frac{\partial (x_1\Phi)}{\partial x_1} + \frac{\partial (x_2\Phi)}{\partial x_2} \right] d\omega + 2\mu\tau \int_\omega \Phi \, d\omega,$$

and applying the Gauss-Ostrogradsky formula to the first integral, we obtain

$$M_t = -\mu\tau\Phi \int_L (x_1 n_1 + x_2 n_2)\, dl + 2\mu\tau \int_\omega \Phi \, d\omega. \tag{7.18}$$

Assuming the constant C in (7.16) to be zero, which is permissible since changing $\Phi(x_1, x_2)$ by a constant leaves the solution of the problem unaltered, as seen from (7.14) and (7.17), instead of formula (7.18) we have

$$M_t = 2\mu\tau \int_\omega \Phi \, d\omega. \tag{7.19}$$

This important formula is due to L. Prandtl.

60. SOME PROPERTIES OF SHEARING STRESSES

We now show that the shearing stress vector $\boldsymbol{T}_3 = \boldsymbol{i}_1\sigma_{31} + \boldsymbol{i}_2\sigma_{32}$ at an arbitrary point M of the cross section of a prismatic body is directed along the tangent to the curve $\Phi(x_1, x_2) =$ constant passing through the point M. Indeed, along the curve $\Phi(x_1, x_2) =$ $=$ constant we have

$$\frac{\partial\Phi}{\partial l} = \frac{\partial\Phi}{\partial x_1}\frac{dx_1}{dl} + \frac{\partial\Phi}{\partial x_2}\frac{dx_2}{dl} = 0.$$

Taking into account formulas (6.27) and (7.14), we find

$$\sigma_{31}n_1 + \sigma_{32}n_2 = \boldsymbol{T}_3 \cdot \boldsymbol{n} = 0,$$

from which $\boldsymbol{T}_3 \perp \boldsymbol{n}$.

Based on what has been proved above, the curves $\Phi(x_1, x_2) =$ $=$ constant are called trajectories or lines of shearing stress. Since $\Phi(x_1, x_2) =$ constant on the contour of the cross section, it is a shearing stress trajectory.

It can easily be proved that both σ_{31} and σ_{32} are harmonic functions in the cross section. Indeed, by applying the harmonic operator Δ to both sides of formulas (7.14), and assuming the legitimacy of interchanging the differential operators, we have, by (7.15),

$$\Delta\sigma_{31} = \mu\tau\Delta\frac{\partial\Phi}{\partial x_2} = \mu\tau\frac{\partial}{\partial x_2}\Delta\Phi = 0,$$

$$\Delta\sigma_{32} = -\mu\tau\frac{\partial}{\partial x_1}\Delta\Phi = 0.$$

It follows that σ_{31} and σ_{32} attain maximum values on the contour of the cross section of a prismatic body.

We now prove that the shearing stress vector $\boldsymbol{T}_3$ also attains its maximum value on the contour; for this we start from the contrary: suppose that the shearing stress vector attains a maximum value inside the contour of the cross section at a point M. We choose a new rectangular Cartesian co-ordinate system $ox_1'x_2'$ at the cross section so that one of its axes, say the ox_2' axis, is directed parallel to the vector $\boldsymbol{T}_3$ applied at the point M. In this co-ordinate system, at the point M we have a stress tensor with components $\sigma_{31}' = 0$, $\sigma_{32}' \neq 0$, and these are also harmonic with respect to the new co-ordinate system. In consequence, σ_{32} attains its maximum value on the contour, and not inside the contour, as was supposed at the beginning of the reasoning.

Hence, the shearing stress vector attains its maximum value on the contour of the cross section of a prismatic body.

61. TORSION OF HOLLOW PRISMATIC BODIES

Let a prismatic body be bounded by several cylindrical surfaces whose axes are parallel. Every cross section of such a bar represents a multiply connected region. In this case the boundary conditions (7.11) take the form

$$\psi = \frac{x_1^2 + x_2^2}{2} + C_\nu \text{ on } L_\nu,$$

where C_ν are constants assuming definite values on each of the contours L_ν the set of which forms the contour of the section.

The torsion function φ must be single valued; otherwise, the displacement $u_3 = \tau\varphi$ would be multiple valued (we are interested in single-valued displacements). The function ψ, conjugate to the single-valued harmonic function and determined from the Cauchy-Riemann conditions (7.10), may, in general, be multiple valued; in our case this must not be so because the function ψ reverts to its original value on passing once round any of the contours L_ν, as seen from the boundary condition for it. Hence, the constants C_ν cannot be fixed in an arbitrary way. Indeed, if they are fixed arbitrarily, and then the function ψ is determined (for this it is necessary to solve the Dirichlet problem, which, as is known, always has a unique solution), the function φ found from the Cauchy-Riemann conditions by means of the function ψ may be multiple valued.

In the present case the function $\Phi(x_1, x_2)$ must, by (7.16), be constant on all contours bounding the section. Thus, the boundary condition for the function $\Phi(x_1, x_2)$ on the contour L_ν is of the form

$$\Phi(x_1, x_2) = C_\nu. \qquad (7.20)$$

As we have seen, the formulas for strains, stresses, and displacements involve partial derivatives of the function Φ. It is therefore sufficient to determine the function $\Phi(x_1, x_2)$ to within an arbitrary constant. This circumstance allows one to set one of the constants C_ν equal to zero.

Let us show that the tangential stresses σ_{31} and σ_{32} at the ends of a prismatic body satisfy the conditions

$$\int_\omega \sigma_{31}\, d\omega = \int_\omega \sigma_{32}\, d\omega = 0 \qquad (7.21)$$

(otherwise, in addition to the applied torque there would be transverse forces at the ends tending to bend the prismatic body).

Inserting expressions (7.14) in the integrals of (7.21), and transforming them into integrals round the contour $L = L_0 + L_1 + L_2 + \ldots + L_m$, we obtain

$$V_1 = \mu\tau \oint_L \Phi n_2 \, dl, \qquad V_2 = -\mu\tau \int_L \Phi n_1 \, dl. \tag{7.22}$$

These integrals may be written as

$$V_1 = \mu\tau \int_{L_0} \Phi n_2 \, dl + \mu\tau \int_{L_1} \Phi n_2 \, dl + \ldots + \mu\tau \int_{L_m} \Phi n_2 \, dl,$$

$$-V_2 = \mu\tau \int_{L_0} \Phi n_1 \, dl + \mu\tau \int_{L_1} \Phi n_1 \, dl + \ldots + \mu\tau \int_{L_m} \Phi n_1 \, dl.$$

With (7.20), we have

$$V_1 = \mu\tau C_0 \int_{L_0} n_2 \, dl + \mu\tau C_1 \int_{L_1} n_2 \, dl + \ldots + \mu\tau C_m \int_{L_m} n_2 \, dl,$$

$$-V_2 = \mu\tau C_0 \int_{L_0} n_1 \, dl + \mu\tau C_1 \int_{L_1} n_1 \, dl + \ldots + \mu\tau C_m \int_{L_m} n_1 \, dl;$$

since

$$\oint_{L_\nu} n_2 \, dl = -\oint_{L_\nu} dx_1 = 0, \qquad \oint_{L_\nu} n_1 \, dl = \oint_{L_\nu} dx_2 = 0,$$

it follows that $V_1 = V_2 = 0$, which was to be proved.

Referring to formulas (7.14) and Fig. 28, we have

$$M_t = -\mu\tau \int_\omega \left(\frac{\partial \Phi}{\partial x_1} x_1 + \frac{\partial \Phi}{\partial x_2} x_2 \right) d\omega =$$

$$= -\mu\tau \int_\omega \left(\frac{\partial (x_1 \Phi)}{\partial x_1} + \frac{\partial (x_2 \Phi)}{\partial x_2} \right) d\omega + 2\mu\tau \int_\omega \Phi \, d\omega. \tag{7.23}$$

By transforming the first integral into a contour one, we have

$$M_t = -\mu\tau \oint_L (x_1 n_1 + x_2 n_2) \Phi \, dl + 2\mu\tau \int_\omega \Phi \, d\omega, \tag{7.24}$$

or, noting that $L = L_0 + L_1 + \ldots + L_m$, we obtain, finally,

$$M_t = -\mu\tau C_0 \int_{L_0} (x_1 n_1 + x_2 n_2) \, dl - \mu\tau C_1 \int_{L_1} (x_1 n_1 + x_2 n_2) \, dl - \ldots$$

$$\ldots - \mu\tau C_n \int_{L_m} (x_1 n_1 + x_2 n_2) \, dl + 2\mu\tau \int_\omega \Phi \, d\omega. \tag{7.25}$$

We transform the contour integrals into surface ones and, remembering that the outer contour of the section must be described in the counter-clockwise direction, and all inner contours in the clockwise direction, rearrange formula (7.25) in the form

$$M_t = 2\mu\tau\left(\sum_{\nu=1}^{m} C_\nu\omega_\nu - C_0\omega_0\right) + 2\mu\tau\int_\omega \Phi\, d\omega, \tag{7.26}$$

where ω_ν is the area bounded by the contour L_ν ($\nu = 0, 1, 2, \ldots$). In the case of a simply connected region we obtain formula (7.19).

62. SHEAR CIRCULATION THEOREM

The circulation of the stress vector $i_1\sigma_{31} + i_2\sigma_{32}$ along a closed line l lying entirely inside the section is

$$\Gamma = \oint_l (i_1\sigma_{31} + i_2\sigma_{32})\cdot dr. \tag{7.27}$$

Inserting (7.2) and (7.1) in the integrand, we have

$$\Gamma = \mu\tau\oint_l (x_1\, dx_2 - x_2\, dx_1) + \mu\oint_l \left(\frac{\partial u_3}{\partial x_1}\, dx_1 + \frac{\partial u_3}{\partial x_2}\, dx_2\right) =$$

$$= 2\mu\tau\omega + \mu\oint_l \frac{\partial u_3}{\partial l}\, dl. \tag{7.28}$$

Since the displacement u_3 must be a single-valued function in the cross section, it follows that

$$\oint_l \frac{\partial u_3}{\partial l}\, dl = 0.$$

By reason of the last circumstance, we find, from (7.28), that

$$\Gamma = 2\mu\tau\omega. \tag{7.29}$$

Here ω is the area bounded by the line of integration.

Formula (7.29) is valid for both simply connected and multiply connected sections, and the line of integration may enclose one, several or none of the inner contours of the section. This formula represents the shear circulation theorem.

The shear circulation may be expressed in terms of the function $\Phi(x_1, x_2)$; for this purpose, we substitute (7.14) in the integrand of (7.27), so that

$$\Gamma = -\mu\tau\oint_l \left(\frac{\partial\Phi}{\partial x_1} n_1 + \frac{\partial\Phi}{\partial x_2} n_2\right) dl = -\mu\tau\oint_l \frac{\partial\Phi}{\partial n}\, dl, \tag{7.30}$$

where n_j is the cosine of the angle between the outward normal to the line of integration and the ox_j axis ($j = 1, 2$).

On comparing formulas (7.29) and (7.30), we obtain

$$-\oint_l \frac{\partial \Phi}{\partial n}\, dl = 2\omega. \tag{7.31}$$

If the line of integration is taken to be an inner contour L_ν, then

$$-\oint_{L_\nu} \frac{\partial \Phi}{\partial n_\nu}\, dl = 2\omega_\nu \quad (\nu = 1, 2, \ldots, m), \tag{7.32}$$

where n_ν is the outward normal to the contour L_ν, ω_ν is the area bounded by the closed contour L_ν. Formulas (7.32) can be used to determine the unknown constants C_ν entering into the boundary conditions (7.20).

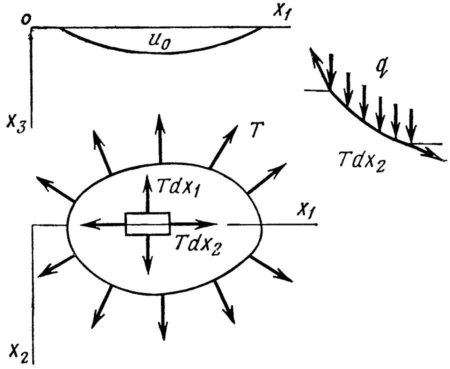


Fig. 29

63. ANALOGIES IN TORSION

(a) *Membrane analogy.* By a membrane is meant a thin film offering no resistance to bending, but acting only in tension.

Suppose that a homogeneous membrane of constant thickness is equally stretched in all directions by a force T over a plane contour of the same shape as the contour of the cross section of a twisted prismatic body, and loaded by a normal uniformly distribued load q per unit area. Let the co-ordinate axes ox_1 and ox_2 lie in the plane of the membrane, which sags under the load q by an amount $u_0(x_1, x_2)$.

Let us derive the differential equation of equilibrium; to do this we cut out an element having the shape of a rectangle of sides dx_1, dx_2 (Fig. 29). Equating to zero the sum of the projections on the ox_3 axis of all forces acting on the element gives

$$-T\frac{\partial u_0}{\partial x_1}\, dx_2 + \left[T\frac{\partial u_0}{\partial x_1}\, dx_2 + \frac{\partial}{\partial x_1}\left(T\frac{\partial u_0}{\partial x_1}\right) dx_1\, dx_2\right] - T\frac{\partial u_0}{\partial x_2}\, dx_1 +$$
$$+\left[T\frac{\partial u_0}{\partial x_2}\, dx_1 + \frac{\partial}{\partial x_2}\left(T\frac{\partial u_0}{\partial x_2}\right) dx_2\, dx_1\right] + q\, dx_1\, dx_2 = 0$$

From this we obtain an equation for the deflection u_0 of a uniformly loaded membrane

$$\frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_2^2} = -\frac{q}{T}. \tag{7.33}$$

Since the deflection u_0 on the contour of the membrane is zero, the contour condition is

$$u_0 = 0. \tag{7.34}$$

Thus, the contour condition (7.34) coincides identically with the contour condition for the function Φ.

On putting $u_0 = k\Phi$, the differential equation (7.15) coincides with (7.33). Inserting $u_0 = k\Phi$ in Eq. (7.33), there results

$$k\left(\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_2^2}\right) = -\frac{q}{T}.$$

On the other hand, we have

$$\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_2^2} = -2.$$

From the last two equations, $k = \frac{q}{2T}$. Then

$$u_0 = \frac{q}{2T}\,\Phi. \tag{7.35}$$

Consequently, the torsion problem for a prismatic body can be solved by measuring the deflections of a uniformly loaded membrane.

If the membrane is cut by planes u_0 = constant, the resulting lines of equal displacement in the torsion problem coincide with the trajectories of shearing stress Φ = constant. The slope of the membrane $\frac{\partial u_0}{\partial n}$ in the direction of the outward normal n to the line of equal displacement at some point determines the shearing stress t at the corresponding point of the section, i.e., $T = \frac{\mu\tau}{k}\frac{\partial u_0}{\partial n}$. Indeed,

$$\frac{\partial u_0}{\partial n} = \frac{\partial k\Phi}{\partial n} = k\left(\frac{\partial\Phi}{\partial x_1} n_1 + \frac{\partial\Phi}{\partial x_2} n_2\right) = -\frac{k}{\mu\tau}(\sigma_{32}n_1 - \sigma_{31}n_2) = \frac{k}{\mu\tau}t,$$

from which

$$t = \frac{\mu\tau}{k}\frac{\partial u_0}{\partial n}.$$

According to this formula, the maximum angle of inclination of the membrane determines the maximum shearing stress.

The torsional rigidity of a prismatic body is determined by the volume v enclosed by the surface of the deformed membrane and the plane of the membrane before deformation, i.e., $D = \frac{2\mu}{k}v$. Indeed,

$$v = \int_\omega u_0\,d\omega = \frac{q}{2T}\int_\omega \Phi\,d\omega;$$

noting that

$$2\mu\tau \int_{\omega} \Phi \, d\omega = M_t \text{ and } M_t = D\tau,$$

we find $v = \frac{q}{4\mu T} D$, from which

$$D = \frac{2\mu}{k} v. \tag{7.36}$$

The value of the membrane analogy resides not only in the fact that it furnishes an experimental means of investigating the torsion problem, but also in the fact that without any experiment the use of this analogy in each specific problem of the torsion of a prismatic body makes it possible to get a qualitative idea of the pattern of shearing stress trajectories and of the maximum tangential stress.

The membrane analogy is easily extended to the case of hollow prismatic bodies. In this case, as is apparent from the relation $u_0 = k\Phi$, which has been derived by comparing Eqs. (7.15) and (7.33), the following conditions must be fulfilled:

(1) the outer contour of the membrane must be similar to the outer contour L_0 of the section of a prismatic body, and must be rigidly fixed;

(2) all inner contours of the section of the prismatic body must be simulated by absolutely rigid plane weightless disks parallel to each other, and must receive translational displacements $u_\nu = kC_\nu$ (C_ν are the constants entering into the boundary conditions on the inner contours of the section of the prismatic body);

(3) these disks must be stressed by the same uniformly distributed normal pressure q as the membrane itself. The last consideration follows from the shear circulation theorem in the torsion problem; the truth of the statement will presently be demonstrated.

Substituting $\Phi = \frac{1}{k} u_0$ in (7.32), we obtain

$$-\oint \frac{\partial u_0}{\partial n_\nu} dl = 2k\omega_\nu = \frac{q}{T} \omega_\nu. \tag{7.37}$$

Here $\frac{\partial u_0}{\partial n_\nu}$ is the slope of the membrane in the direction of the outward normal $\boldsymbol{n}_\nu$ to the inner contour L_ν, ω_ν is the area bounded by the inner contour L_ν.

By multiplying both sides of (7.37) by the amount of uniform tension in the membrane T, we have

$$-\oint T \frac{\partial u_0}{\partial n_\nu} dl = q\omega_\nu. \tag{7.38}$$

Obviously, the left-hand side of this equality is the sum of the projections of the forces of tension in the membrane at the section through the given contour L_ν on a direction perpendicular to the plane of the contour L_ν.

Thus, (7.38) gives a condition for the equilibrium of each disk under the uniformly distributed pressure and the tension in the membrane at the section through the contour of this disk. If this kind of membrane together with a disk is stressed by a uniform pressure, we obtain the membrane analogy of the torsion problem for a prismatic body of multiply connected section (Fig. 30), i.e., the displacement of the membrane is proportional to the function Φ (x_1, x_2), and the lines of equal displacement are similar to the shearing stress trajectories.

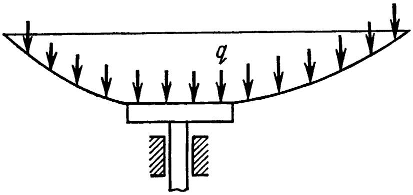


Fig. 30

The realization of the membrane analogy experiment in the case of the torsion problem for a prismatic body of multiply connected section presents great difficulties. However, for a qualitative study of a specific problem of the torsion of a hollow prismatic body, as has already been mentioned in the case of simply connected regions, the membrane analogy is of great value.

As an example let us consider the problem of the torsion of thin-walled tubes.

To investigate the torsion of thin-walled tubes using the membrane analogy, it is necessary to fix the membrane along its contour, which must be similar to the outer contour of the section, and superimpose an absolutely rigid plane disk having the shape of the inner contour. Next, the membrane and disk must be stressed by a uniformly distributed pressure giving the disk a translational motion in a direction perpendicular to its plane (Fig. 30). Since we are considering the case when the wall thickness of the tube is small, the deformation of the membrane is determined mainly by the load exerted on the disk; as regards the load acting directly on the membrane, it may be neglected. It appears from the above that the surface of the deformed membrane coincides closely with a conic surface connecting both contours. This conclusion allows an approximate analysis to be made in the study of the torsion of thin-walled tubes of arbitrary cross section.

Inside a ring section draw a line L equidistant from both its boundaries (Fig. 31) and take some point A on this line as the origin of arc length l. The ring section is specified if the line L and the thickness $\delta = \delta\,(l)$ are known. The value of the shearing stress T at a given point can be approximately estimated from the mean slope of the membrane at this point; hence, the approximate value of the tangential stress at the point B is determined by the formula

$$T = \mu\tau\frac{C_1 - C_0}{\delta\,(l)}.$$

Assuming that $C_0 = 0$ on the outer contour, we have

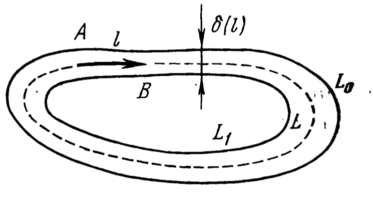
$$T = \mu\tau\frac{C_1}{\delta\,(l)}. \qquad (7.39)$$

As seen from this formula,

$$T_{\max} = \mu\tau\frac{C_1}{\delta_{\min}}. \qquad (7.40)$$

From formula (7.26) we have

$$M_t = 2\mu\tau\left(C_1\omega_1 + \int\limits_{\omega} \Phi\, d\omega\right),$$



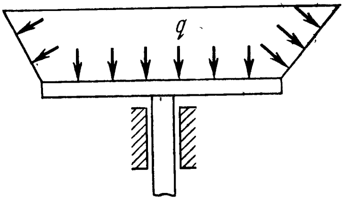


Fig. 31

where ω_1 is the area bounded by the inner contour, C_1 is the constant value of the function Φ on the inner contour.

Noting that the mean value of the function Φ on the line L is approximately equal to $1/2\ C_1$, the last formula is rearranged in the form

$$M_t = 2\mu\tau C_1\left(\omega_1 + \frac{1}{2}\int\limits_{\omega} d\omega\right) = 2\mu\tau C_1\left(\omega_1 + \frac{1}{2}\int\limits_{L} \delta\,(l)\,dl\right). \qquad (7.41)$$

The expression within the parentheses on the right-hand side of (7.41) represents the area bounded by the mean contour L; hence, formula (7.41) is rewritten as

$$M_t = 2\mu\tau C_1\omega, \qquad (7.42)$$

where

$$\omega = \omega_1 + \frac{1}{2}\int\limits_{L} \delta\,(l)\,dl.$$

On the other hand, we have, by (7.32),

$$-\oint\limits_{L} \frac{0 - C_1}{\delta\,(l)}\,dl = 2\omega. \qquad (7.43)$$

By eliminating C_1 from formulas (7.42) and (7.43), we obtain

$$M_t = \frac{4\mu\tau\omega^2}{I}, \tag{7.44}$$

where

$$I = \int_L \frac{dl}{\delta(l)}.$$

Formula (7.44) is given by R. Bredt.

On comparing formulas (7.39) and (7.42), we find

$$T = \frac{M_t}{2\omega\delta(l)}.$$

This formula is also due to R. Bredt.

(b) *Boussinesq's hydrodynamic analogy.* Consider the laminar motion of a viscous fluid through a prismatic tube with a cross section coinciding with the cross section of a prismatic body whose torsion is under investigation. Let the axis of the tube be denoted by ox_3. The velocity $v(x_1, x_2)$ of the fluid flowing through the tube must satisfy Poisson's equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = \frac{1}{\mu_0}\frac{\partial p}{\partial x_3}, \tag{7.45}$$

where $\frac{\partial p}{\partial x_3}$ is the drop of hydrodynamic pressure along the axis of the tube, which is taken to be constant.

On the walls of the tube we have the Reynolds condition

$$v = 0. \tag{7.46}$$

Thus, the contour condition (7.46) coincides identically with the contour condition for the function Φ.

On putting $v = k\Phi$, the differential equation (7.15) coincides with (7.45). Inserting $v = k\Phi$ in Eq. (7.45), and comparing the resulting equation with Eq. (7.15), we obtain

$$k = \frac{-\frac{\partial p}{\partial x_3}}{2\mu_0}$$

and

$$v = \frac{-\frac{\partial p}{\partial x_3}}{2\mu_0}\Phi.$$

64. COMPLEX TORSION FUNCTION

It is often convenient in the solution of the torsion problem to introduce a function $F(z)$ of the complex variable $z = x_1 + ix_2$, related to the torsion function $\varphi(x_1, x_2)$ and its conjugate $\psi(x_1, x_2)$, in the form

$$iF(z) = \varphi + i\psi. \tag{7.47}$$

The function $F(z)$ is obviously holomorphic in the region occupied by any cross section of the body.

By (7.11), for a multiply connected region the boundary conditions that must be satisfied by the function $F(z)$ become

$$F(t) + \overline{F(t)} = t\bar{t} + C_\nu, \tag{7.48}$$

where t is the affix of a point; C_ν are constants, one of which may be fixed arbitrarily, and the others are to be determined.

In the case of a simply connected region we have

$$F(t) + \overline{F(t)} = t\bar{t} + C. \tag{7.49}$$

On the basis of formulas (7.2),

$$\sigma_{31} - i\sigma_{32} = \mu\tau\left(\frac{\partial\varphi}{\partial x_1} - i\frac{\partial\varphi}{\partial x_2} - x_2 - ix_1\right),$$

from which, remembering that $\frac{\partial\varphi}{\partial x_2} = -\frac{\partial\psi}{\partial x_1}$, we obtain, finally,

$$\sigma_{31} - i\sigma_{32} = \mu\tau i\,[F'(z) - \bar{z}]. \tag{7.50}$$

The torsional rigidity of a prismatic body is determined by formula (7.9), which is rearranged in the form

$$D = \mu \int_\omega (x_1^2 + x_2^2)\,d\omega - \mu \int_\omega \left[\frac{\partial(x_2\varphi)}{\partial x_1} - \frac{\partial(x_1\varphi)}{\partial x_2}\right] d\omega.$$

By applying the Gauss-Ostrogradsky formula to the second integral, and introducing the notation $I_0 = \int_\omega (x_1^2 + x_2^2)\,d\omega$, we obtain

$$D = \mu\left[I_0 - \int_L \varphi\,(x_1\,dx_1 + x_2\,dx_2)\right].$$

Here I_0 is the polar moment of inertia of the cross-sectional area. By using formula (7.47), and taking into account that $x_1dx_1 + x_2dx_2 = \frac{1}{2}d(t\bar{t})$, the last formula may be put into the form:

$$D = \mu\left[I_0 - \frac{i}{4}\int_L \left(F(t) - \overline{F(t)}\right) d(t\bar{t})\right].$$

If the cross section of a prismatic body represents a multiply connected region, the last formula becomes

$$D = \mu\left[I_0 - \frac{i}{4}\sum_{j=0}^{m}\int_{L_j} (F(t) - \overline{F(t)})\,d(t\bar{t})\right] \tag{7.51}$$

Here the integration is carried out along all contours $L_0, L_1, \ldots, L_m$, the sense of description of which is such that the region remains on the left.

65. SOLUTION OF SPECIAL TORSION PROBLEMS

Below are given several examples of the solution of special torsion problems for prismatic bars.

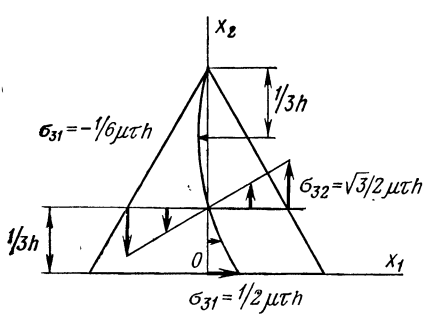

x_2
$1/3h$
$\sigma_{31}=-1/6\mu\tau h$
$\sigma_{32}=\sqrt{3}/2\mu\tau h$
$1/3h$
0
x_1
$\sigma_{31}=1/2\mu\tau h$

Fig. 32

(a) *Prismatic bar of elliptical section.* Prandtl's stress function $\Phi(x_1, x_2)$ must be constant on the ellipse

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1.$$

The function $\Phi(x_1, x_2)$ satisfying the boundary condition may be represented as

$$\Phi(x_1, x_2)=A\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\right), \quad (7.52)$$

where A is an unknown constant. Moreover, the function $\Phi(x_1, x_2)$ must satisfy Poisson's equation inside the ellipse; hence, for the determination of the value of A we obtain the relation

$$2A\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=-2,$$

from which we find

$$A=-\frac{a^2b^2}{a^2+b^2}.$$

Then

$$\Phi(x_1, x_2)=-\frac{b^2x_1^2+a^2x_2^2}{a^2+b^2}. \quad (7.53)$$

Substituting (7.53) in relations (7.14), the stresses σ_{31} and σ_{32} are obtained as

$$\sigma_{31}=-\frac{2a^2\mu\tau}{a^2+b^2}x_2,$$

$$\sigma_{32}=\frac{2b^2\mu\tau}{a^2+b^2}x_1.$$

(b) *Prismatic bar whose section is an equilateral triangle* (Fig. 32). The equations of the sides of an equilateral triangle of height h are:

$$x_2=0, \quad x_2=-\sqrt{3}x_1+h, \quad x_2=\sqrt{3}x_1+h.$$

The function $\Phi(x_1, x_2)$ must be constant on these sides. This, as in the first problem, suggests that the function should be represented in the form:

$$\Phi(x_1, x_2) = Ax_2\,[(x_2 - h) + \sqrt{3}x_1]\,[(x_2 - h) - \sqrt{3}x_1]. \tag{7.54}$$

This function is zero on the sides of the triangle. Moreover, the function $\Phi(x_1, x_2)$ must satisfy Poisson's equation inside the triangle.

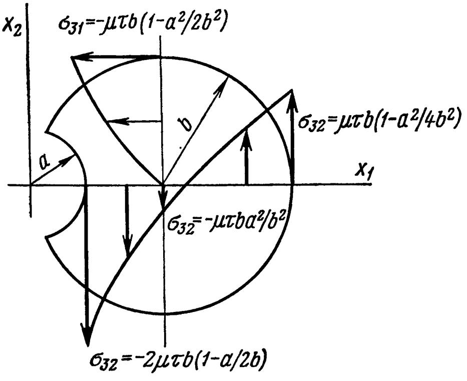

x_2
$\sigma_{31} = -\mu\tau b(1 - a^2/2b^2)$
b
$\sigma_{32} = \mu\tau b(1 - a^2/4b^2)$
a
x_1
$\sigma_{32} = -\mu\tau b a^2/b^2$
$\sigma_{32} = -2\mu\tau b(1 - a/2b)$

Fig. 33

From this we find that $A = \frac{1}{2h}$. We obtain, finally,

$$\Phi(x_1, x_2) = \frac{1}{2h}\,x_2\,[(x_2 - h)^2 - 3x_1^2].$$

From formulas (7.14) we find

$$\sigma_{13} = \frac{\mu\tau}{2h}\,[3\,(x_2^2 - x_1^2) + h\,(h - 4x_2)], \qquad \sigma_{23} = \frac{3\mu\tau}{h}\,x_1x_2.$$

(c) *Circular prismatic bar with a semicircular longitudinal groove* (Fig. 33). The equation of the contour of the section is

$$(x_1 - b)^2 + x_2^2 - b^2 = 0, \qquad x_1^2 + x_2^2 - a^2 = 0.$$

The stress function, which must be zero on the contour, is sought in the form

$$\Phi(x_1, x_2) = \frac{A}{x_1^2 + x_2^2}\,[(x_1 - b)^2 + x_2^2 - b^2]\,(x_1^2 + x_2^2 - a^2)$$

or

$$\Phi(x_1, x_2) = A\left(x_1^2 + x_2^2 - a^2 - 2bx_1 + \frac{2ba^2x_1}{x_1^2 + x_2^2}\right).$$

In addition, the function $\Phi(x_1, x_2)$ must satisfy Eq. (7.15) inside the above contour. Hence, we find that $A = -1/2$. Consequently,

$$\Phi(x_1, x_2) = -\frac{1}{2}\left[x_1^2 + x_2^2 - a^2 - 2bx_1\left(\frac{a^2}{x_1^2+x_2^2} - 1\right)\right].$$

According to formulas (7.14), we have

$$\sigma_{31} = -\mu\tau\left[1 - \frac{2ba^2x_1}{(x_1^2+x_2^2)^2}\right]x_2,$$

$$\sigma_{32} = \mu\tau\left[(x_1 - b) + \frac{a^2b}{x_1^2+x_2^2}\left(1 - \frac{2x_1^2}{x_1^2+x_2^2}\right)\right].$$

(d) *Prismatic bar with a section in the form of a rectangle.* In order to find the solution of the torsion problem for the indicated bar, we determine a harmonic function $\psi(x_1, x_2)$ that assumes the value $\frac{1}{2}(x_1^2 + x_2^2)$ on the boundaries $x_1 = \pm a$, $x_2 = \pm b$ of the rectangle.

The unknown function $\psi(x_1, x_2)$ is represented as the sum of two harmonic functions

$$a^2 + \frac{1}{2}(x_2^2 - x_1^2) \text{ and } q(x_1, x_2),$$

i.e.,

$$\psi(x_1, x_2) = a^2 + \frac{1}{2}(x_2^2 - x_1^2) + q(x_1, x_2). \tag{7.55}$$

The newly introduced harmonic function $q(x_1, x_2)$ must satisfy on the boundary of the rectangle the conditions

$$q(\pm a, x_2) = 0, \quad q(x_1, \pm b) = x_1^2 - a^2. \tag{7.56}$$

The harmonic function $q(x_1, x_2)$ is taken in the form of a series

$$q(x_1, x_2) = \sum_{n=0}^{\infty} \alpha_n \cosh(\beta_n x_2)\cos(\beta_n x_1). \tag{7.57}$$

By using (7.57) in the boundary conditions (7.56), we obtain

$$\beta_n = \left(n + \frac{1}{2}\right)\frac{\pi}{a},$$

$$x_1^2 - a^2 = \sum_{n=0}^{\infty} \alpha_n \cosh(\beta_n b)\cos(\beta_n x_1).$$

From the second relation we find

$$\alpha_n = \frac{1}{a\cosh(\beta_n b)}\int_{-a}^{a}(x_1^2 - a^2)\cos(\beta_n x_1)\,dx_1 = -\frac{4}{a\beta_n^3}\frac{\sin(\beta_n a)}{\cosh(\beta_n b)}.$$

The result is

$$\psi(x_1, x_2) = a^2 + \frac{1}{2}(x_2^2 - x_1^2) -$$

$$- \frac{32a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \, \frac{\cosh (2n+1) \frac{\pi}{2a} x_2}{\cosh (2n+1) \frac{\pi}{2a} b} \cos (2n+1) \frac{\pi}{2a} x_1 .$$

The components of the stress tensor can now be calculated by formulas (7.12).

(e) *Prismatic body of circular section with a circular eccentric hole.* Denote by S the region occupied by any one cross section of the body, bounded from the outside by a circumference L_0 of radius R, and from the inside by a circumference L_1 of radius r; the affix of the centre of the latter circumference is designated as e (Fig. 34). For the present case the boundary conditions (7.48) become

$$F(t) + \overline{F(t)} = t\bar{t} + C_0 \text{ on } L_0, \quad (7.58)$$

$$F(t) + \overline{F(t)} = t\bar{t} + C_1 \text{ on } L_1. \quad (7.59)$$

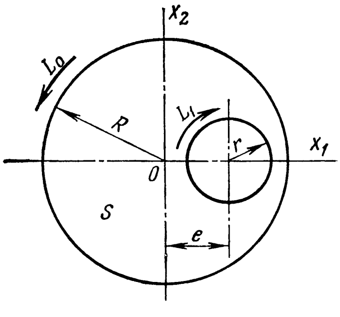


Fig. 34

Since $t\bar{t} = R^2$ on L_0 and the constant C_0 can be chosen equal to $-R^2$, condition (7.58) is written as

$$F(t) + \overline{F(t)} = 0 \text{ on } L_0; \quad (7.60)$$

the expression for $t\bar{t}$ on L_1 may be written as follows:

$$t\bar{t} = (e + re^{i\theta})(e + re^{-i\theta}) = er\left\{\frac{t-e}{r} + \frac{r}{t-e}\right\} + e^2 + r^2;$$

hence, we have, on L_1,

$$F(t) + \overline{F(t)} = er\left\{\frac{t-e}{r} + \frac{r}{t-e}\right\} + d_0. \quad (7.61)$$

Here $d_0 = e^2 + r^2 + C_1$.

The solution of the problem is sought in the form

$$F(z) = \sum_{k=0}^{\infty} a_k \left(\frac{z}{R}\right)^k + \sum_{k=1}^{\infty} b_k \left(\frac{r}{z-e}\right)^k, \quad (7.62)$$

where the first series represents a holomorphic function inside L_0, and the second series represents a holomorphic function outside L_1. The coefficients a_k and b_k are assumed to be real.

Substituting the boundary values of the function $F(z)$ in conditions (7.58) and (7.59), we obtain

$$\sum_{k=0}^{\infty} a_k \left(\frac{t}{R}\right)^k + \sum_{k=1}^{\infty} b_k \left(\frac{r}{t-e}\right)^k + \sum_{k=0}^{\infty} a_k \left(\frac{R}{t}\right)^k + \sum_{k=1}^{\infty} b_k \left(\frac{r}{\bar{t}-e}\right)^k = 0, \tag{7.63}$$

$$\sum_{k=0}^{\infty} a_k \left(\frac{t}{R}\right)^k + \sum_{k=1}^{\infty} b_k \left(\frac{r}{t-e}\right)^k + \sum_{k=0}^{\infty} a_k \left(\frac{\bar{t}}{R}\right)^k + \sum_{k=1}^{\infty} b_k \left(\frac{r}{\bar{t}-e}\right)^k = er\left\{\frac{t-e}{r} + \frac{r}{t-e}\right\} + d_0. \tag{7.64}$$

We now transform the second term in condition (7.63)

$$\sum_{k=1}^{\infty} b_k \left(\frac{r}{t-e}\right)^k = \sum_{k=1}^{\infty} b_k \left(\frac{r}{R}\right)^k \left(\frac{R}{t}\right)^k \frac{1}{\left(1-\frac{e}{t}\right)^k} =$$

$$= \sum_{k=1}^{\infty} b_k \left(\frac{r}{R}\right)^k \left(\frac{R}{t}\right)^k \sum_{\nu=0}^{\infty} (-1)^{\nu} C_{-k}^{\nu} \left(\frac{e}{t}\right)^{\nu} =$$

$$= \sum_{k=1}^{\infty} b_k \left(\frac{r}{R}\right)^k \sum_{\nu=0}^{\infty} (-1)^{\nu} C_{-k}^{\nu} \left(\frac{e}{R}\right)^{\nu} \left(\frac{R}{t}\right)^{k+\nu},$$

where $C_{-k}^{\nu} = (-1)^{\nu} C_{k+\nu-1}^{\nu}$.
Introducing a new index $n = k + \nu$ instead of the index ν gives

$$\sum_{k=1}^{\infty} b_k \left(\frac{r}{t-e}\right)^k = \sum_{k=1}^{\infty} b_k \left(\frac{r}{R}\right)^k \sum_{n=k}^{\infty} (-1)^{n-k} C_{-k}^{n-k} \left(\frac{e}{R}\right)^{n-k} \left(\frac{R}{t}\right)^n.$$

Here the double summation is carried out over the whole-numbered points of the angle ABC (Fig. 35).

By interchanging the order of summation in the last double sum, we find

$$\sum_{k=1}^{\infty} b_k \left(\frac{r}{t-e}\right)^k = \sum_{n=1}^{\infty} B_n \left(\frac{R}{t}\right)^n, \tag{7.65}$$

where

$$B_n = \sum_{k=1}^{n} (-1)^{n-k} \left(\frac{r}{R}\right)^k \left(\frac{e}{R}\right)^{n-k} C_{-k}^{n-k} b_k. \tag{7.66}$$

Obviously,

$$\sum_{k=1}^{\infty} b_k \left(\frac{r}{\bar{t}-e}\right)^k = \sum_{k=1}^{\infty} \overline{b_k \left(\frac{r}{t-e}\right)^k} = \sum_{n=1}^{\infty} B_n \left(\frac{t}{R}\right)^n. \tag{7.67}$$

Inserting expansion (7.65) in condition (7.63), we obtain $a_0 = 0$ and an infinite system of linear equations

$$a_n + B_n = 0 \quad (n = 1, 2, \ldots). \tag{7.68}$$

To obtain a second infinite system of linear equations, we transform the first term in condition (7.64):

$$\sum_{k=0}^{\infty} a_k \left(\frac{t}{R}\right)^k = \sum_{k=0}^{\infty} a_k \left(\frac{t-e}{R} + \frac{e}{R}\right)^k =$$

$$= \sum_{k=0}^{\infty} a_k \sum_{n=0}^{k} C_k^n \left(\frac{r}{R}\right)^n \left(\frac{e}{R}\right)^{k-n} \left(\frac{t-e}{r}\right)^n.$$

Here the double summation is carried out over the whole-numbered

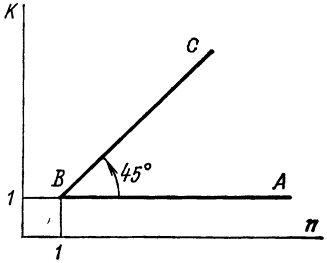


Fig. 35

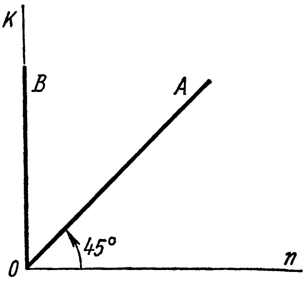


Fig. 36

points of the angle *AOB* (Fig. 36). By interchanging the order of summation, we have

$$\sum_{k=0}^{\infty} a_k \left(\frac{t}{R}\right)^k = \sum_{n=0}^{\infty} A_n \left(\frac{t-e}{r}\right)^n, \tag{7.69}$$

where

$$A_n = \left(\frac{r}{R}\right)^n \sum_{k=0}^{\infty} C_k^n \left(\frac{e}{R}\right)^{k-n} a_k. \tag{7.70}$$

Obviously,

$$\sum_{k=0}^{\infty} a_k \left(\frac{\bar{t}}{R}\right)^k = \sum_{n=0}^{\infty} A_n \left(\frac{r}{t-e}\right)^n. \tag{7.71}$$

Condition (7.64) then becomes

$$\sum_{n=1}^{\infty}\left\{(A_n+b_n)\left[\left(\frac{t-e}{r}\right)^n+\left(\frac{r}{t-e}\right)^n\right]\right\}+2A_0=er\left\{\frac{t-e}{r}+\frac{r}{t-e}\right\}+d_0.$$

Hence,

$$A_0=\frac{d_0}{2}\text{ or }\sum_{k=1}^{\infty}\left(\frac{e}{R}\right)^k a_k=\frac{d_0}{2}, \tag{7.72}$$

$$A_n+b_n=er\varepsilon_n \quad (n=1,\ 2,\ 3,\ \ldots), \tag{7.73}$$

where $\varepsilon_1 = 1$ and $\varepsilon_n = 0$ for $n = 2, 3, 4, \ldots$.

On the basis of the theorem of existence and uniqueness of solution of elasticity problems we may draw a conclusion that the set of two infinite systems of linear equations (7.68) and (7.73) has a solution, and, moreover, the solution is unique and bounded; its approximate solution is the solution of two finite systems, and the number of these equations must be fixed according to the parameter defining the closeness of the contours of the section to one another and the required accuracy of the analysis. After determining the roots of Eqs. (7.68) and (7.73), the constant C_1 is found from relation (7.72).

As an illustration of the solution, consider a numerical example. We take the first three equations of (7.68) and (7.73):

$$\begin{gathered}
a_1+\frac{r}{R}b_1=0,\\
a_2+\frac{r}{R}\frac{e}{R}\,b_1+\left(\frac{r}{R}\right)^2 b_2=0,\\
a_3+\frac{r}{R}\left(\frac{e}{R}\right)^2 b_1+2\left(\frac{r}{R}\right)^2\frac{e}{R}\,b_2+\left(\frac{r}{R}\right)^3 b_3=0,\\
\frac{r}{R}a_1+2\frac{r}{R}\frac{e}{R}a_2+3\frac{r}{R}\left(\frac{e}{R}\right)^2 a_3+b_1=\frac{e}{R}\frac{r}{R}R^2,\\
\left(\frac{r}{R}\right)^2 a_2+3\left(\frac{r}{R}\right)^2\frac{e}{R}a_3+b_2=0,\\
\left(\frac{r}{R}\right)^3 a_3+b_3=0.
\end{gathered} \tag{7.74}$$

From the first three equations of this system we have

$$b_1=-\frac{R}{r}a_1,\quad b_2=-\left(\frac{R}{r}\right)^2\left(a_2-\frac{e}{R}a_1\right),$$

$$b_3=-\left(\frac{R}{r}\right)^3\left\{a_3-\left(\frac{e}{R}\right)^2 a_1-2\frac{e}{R}\left(a_2-\frac{e}{R}a_1\right)\right\};$$

the last three equations of system (7.74) then become

$$\left(\frac{r}{R}-\frac{R}{r}\right)a_1+2\frac{r}{R}\frac{e}{R}a_2+3\frac{r}{R}\left(\frac{e}{R}\right)^2a_3=\frac{e}{R}\frac{r}{R}R^2,$$

$$\frac{e}{R}\left(\frac{R}{r}\right)^2a_1+\left[\left(\frac{r}{R}\right)^2-\left(\frac{R}{r}\right)^2\right]a_2+3\left(\frac{r}{R}\right)^2\frac{e}{R}a_3=0,$$

$$-\left(\frac{R}{r}\right)^3\left(\frac{e}{R}\right)^2a_1+2\left(\frac{R}{r}\right)^3\frac{e}{R}a_2+\left[\left(\frac{r}{R}\right)^3-\left(\frac{R}{r}\right)^3\right]a_3=0.$$

For the relative dimensions $r/R = 0.2$, $e/R = 0.6$ the roots are:

$$a_1 = -0.262167 \times 10^{-1}R^2, \quad a_2 = -0.157831 \times 10^{-1}R^2,$$
$$a_3 = -0.250176 \times 10^{-2}R^2, \quad b_1 = 0.131084R^2,$$
$$b_2 = 0.125 \times 10^{-2}R^2, \quad b_3 = 0.125 \times 10^{-4}R^2.$$

In the example under consideration, from (7.72) we have $C_1 = -0.446929R^2$.

The values of

$$\Delta=\frac{F(t)+\overline{F(t)}}{t\bar t+C_0}100\%$$

for the points $t = R$, $t = iR$, $t = -R$ of the circumference L_0, and the values of

$$\Delta=\frac{F(t)+\overline{F(t)}-(t\bar t+C_1)}{t\bar t+C_1}100\%$$

for the points $t = e + r$ and $t = e - r$ of the circumference L_1 are, respectively, 1.490%, 1.012%, −0.249%, 0.132%, −6.7%. It appears from these figures that the boundary conditions are fulfilled with reasonable accuracy; hence, the solution is quite efficient.

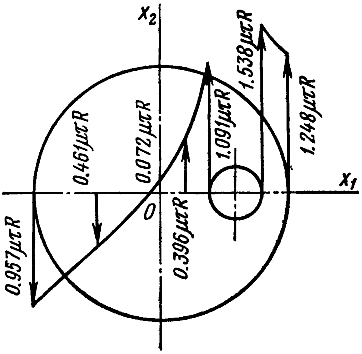


Fig. 37

On the basis of formula (7.50) the values of shearing stresses are calculated at points of the x_1 axis, and the shearing stress diagram is constructed (Fig. 37). As seen from the diagram, the disturbance introduced by the hole is of a local nature.

We now determine the rigidity. Taking into account that $d(t\bar t) = dR^2 = 0$, $d(t\bar t) = \text{red}\left(\frac{r}{t-e}+\frac{t-e}{r}\right)$, respectively, on L_0 and L_1, from formula (7.51) we obtain

$$D=\mu\frac{\pi R^4}{2}\left[1-\left(\frac{r}{R}\right)^4+2\left(\frac{e}{R}\right)^2\left(\frac{r}{R}\right)^2\right]-$$
$$-\frac{i}{4}\mu\oint\limits_{L_1}\sum_{k=1}^{\infty}\left\{a_k\left[\left(\frac{t}{R}\right)^k-\left(\frac{\bar t}{R}\right)^k\right]+b_k\left[\left(\frac{r}{t-e}\right)^k-\left(\frac{t-e}{r}\right)^k\right]\right\}\times$$
$$\times d\left(\frac{r}{t-e}+\frac{t-e}{r}\right)\frac{r}{R}\frac{e}{R}R^2.$$

By using formula (7.69) and Cauchy's theorem, we obtain, finally,

$$D=\mu\frac{\pi R^4}{2}\left[1-\left(\frac{r}{R}\right)^4+2\left(\frac{e}{R}\right)^2+\left(\frac{r}{R}\right)^2\right]+\pi(A_1-b_1)\frac{r}{R}\frac{e}{R}R^2\mu.$$

In the example under consideration $D = 1.559\mu R^4$; hence,

$$\tau=\frac{M_t}{D}=0.641\frac{M_t}{\mu R^4}.$$

Here M_t is the given twisting moment.

66. BENDING OF A PRISMATIC BODY FIXED AT ONE END

Suppose that a prismatic body of length l is fixed at one end and carries at the free end a load statically equivalent to a force $\boldsymbol{P}$ perpendicular to the axis of the body. Body forces and forces on the lateral surface of the body are absent. Let the origin be placed at an arbitrary point of any one section, with the ox_3 axis directed parallel to the axis of the body, and the ox_1 axis parallel to the force $\boldsymbol{P}$ (Fig. 38). The section is assumed to be simply connected.

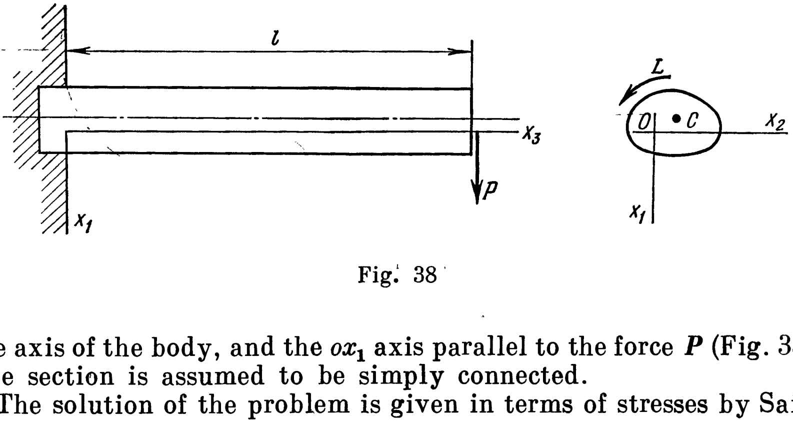


Fig. 38

The solution of the problem is given in terms of stresses by Saint Venant's semi-inverse method. From physical considerations we assume

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \tag{7.75}$$

$$\sigma_{33} = P\,(ax_1 + bx_2 + e)\,(l - x_3); \tag{7.76}$$

the components σ_{31} and σ_{32} of the stress tensor are to be determined. It will be shown below that the coefficients a, b, e are uniquely determined by the shape and dimensions of the cross section of the body and by the choice of co-ordinate system.

The components of the stress tensor σ_{33}, σ_{31}, and σ_{32} at any section x_3 in the present problem must satisfy the equilibrium conditionstions:

$$\int_\omega \sigma_{31}d\omega - P = 0,\quad \int_\omega \sigma_{32}d\omega = 0,\quad \int_\omega (x_1\sigma_{32} - x_2\sigma_{31})\,d\omega = 0, \qquad (7.77)$$

$$\int_\omega \sigma_{33}d\omega = 0,\quad \int_\omega \sigma_{33}x_2d\omega = 0,\quad \int_\omega \sigma_{33}x_1d\omega + P(l - x_3) = 0. \qquad (7.78)$$

Substituting expression (7.76) in conditions (7.78), we obtain a system of three linear equations for the coefficients a, b, and e:

$$\begin{aligned} aS_2 + bS_1 + e\omega &= 0,\\ aI_{12} + bI_{11} + eS_1 &= 0,\\ aI_{22} + bI_{12} + eS_2 &= -1, \end{aligned} \qquad (7.79)$$

where I_{11}, I_{22}, I_{12}, S_1, and S_2 are the moments of inertia and the static moments of the cross-sectional area of the body with respect to the x_1 and x_2 axes, and ω is the cross-sectional area.

The roots of system (7.79) are

$$a = -\frac{I_{11}\omega - S_1^2}{B},\quad b = \frac{I_{12}\omega - S_1S_2}{B},\quad e = \frac{I_{11}S_2 - I_{12}S_1}{B}, \qquad (7.80)$$

where

$$B = \begin{vmatrix} I_{22} & I_{12} & S_2 \\ I_{12} & I_{11} & S_1 \\ S_2 & S_1 & \omega \end{vmatrix}.$$

On the basis of the formulas $S_1 = \omega x_{2c}$, $S_2 = \omega x_{1c}$ the formula for the coefficient e is transformed into

$$e = -ax_{1c} - bx_{2c}. \qquad (7.81)$$

Here x_{1c}, x_{2c} are the co-ordinates of the centroid of the cross-sectional area.

Substituting (7.75) and (7.76) in the differential equations of equilibrium, with $F_j = 0$, we obtain

$$\frac{\partial\sigma_{31}}{\partial x_3} = 0,\quad \frac{\partial\sigma_{32}}{\partial x_3} = 0, \qquad (7.82)$$

$$\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} - P(ax_1 + bx_2 + e) = 0. \qquad (7.83)$$

It follows from (7.82) that σ_{31} and σ_{32} are independent of the x_3 coordinate; hence, they are distributed in the same manner at all cross sections.

Equation (7.83) is given a new representation

$$\frac{\partial}{\partial x_1}\left[\sigma_{31}-\frac{1}{2}P(ax_1^2+ex_1)\right]+\frac{\partial}{\partial x_2}\left[\sigma_{32}-\frac{1}{2}P(bx_2^2+ex_2)\right]=0. \quad (7.84)$$

It follows from this equation that there exists a function $\chi(x_1, x_2)$ related to σ_{31} and σ_{32} by the equalities

$$\begin{aligned}\sigma_{31}&=\frac{P}{2}\left(\frac{\partial\chi}{\partial x_2}+ax_1^2+ex_1\right),\\ \sigma_{32}&=\frac{P}{2}\left(-\frac{\partial\chi}{\partial x_1}+bx_2^2+ex_2\right).\end{aligned} \quad (7.85)$$

Indeed, on substituting (7.85) in equality (7.84) the latter is satisfied identically.

We now derive the conditions that must be satisfied by the function $\chi(x_1, x_2)$. For this, let (7.75), (7.76), and (7.85) be subject to the Beltrami-Michell relations and the boundary condition on the lateral surface of the body.

Of the six Beltrami-Michell relations four relations are satisfied identically, and two relations lead to the equations

$$\begin{aligned}\frac{\partial}{\partial x_2}(\Delta\chi)&=-\frac{2\nu}{1+\nu}a,\\ \frac{\partial}{\partial x_1}(\Delta\chi)&=\frac{2\nu}{1+\nu}b,\end{aligned}$$

from which

$$d(\Delta\chi)=\left(-\frac{2\nu}{1+\nu}a\right)dx_2+\frac{2\nu}{1+\nu}bdx_1.$$

Then

$$\Delta\chi=\frac{2\nu}{1+\nu}(bx_1-ax_2)-2C. \quad (7.86)$$

Here C is an integration constant to be determined.

The conditions of zero load on the lateral surface of the body give, in the present case,

$$\sigma_{31}n_1+\sigma_{32}n_2=0 \text{ on } L. \quad (7.87)$$

Taking into account formulas (7.85), with (6.27), from (7.87) we have the boundary condition for the function $\chi(x_1, x_2)$ on L

$$\frac{\partial\chi}{\partial l}=(bx_2^2+ex_2)\frac{dx_1}{dl}-(ax_1^2+ex_1)\frac{dx_2}{dl}. \quad (7.88)$$

Next, we replace problem (7.86), (7.88) by two problems; for this, the function $\chi(x_1, x_2)$ is represented as follows:

$$\chi=\Psi+C\Phi, \quad (7.89)$$

where Ψ and Φ are some new functions to be determined.

Substituting the last relation in (7.86) and (7.88), problem (7.86), (7.88) is broken down into the following two problems:

$$\begin{gathered} \Delta\Phi = -2, \\ \frac{\partial\Phi}{\partial l} = 0 \text{ on } L, \end{gathered} \tag{7.90}$$

$$\begin{gathered} \Delta\Psi = \frac{2\nu}{1+\nu}(bx_1 - ax_2), \\ \frac{\partial\Psi}{\partial l} = (bx_2^2 + ex_2)\frac{dx_1}{dl} - (ax_1^2 + ex_1)\frac{dx_2}{dl} \text{ on } L. \end{gathered} \tag{7.91}$$

Consequently, the function Φ is Prandtl's stress function.

Thus, problem (7.86), (7.88) of the transverse bending of a prismatic body is divided into the torsion problem (7.90) and problem (7.91) of finding an auxiliary function Ψ called the flexure function.

For simply connected cross sections the boundary conditions on L reduce to

$$\Phi = 0, \tag{7.92}$$

$$\Psi = \int_0^l \left[(bx_2^2 + ex_2)\frac{dx_1}{dl} - (ax_1^2 + ex_1)\frac{dx_2}{dl}\right] dl. \tag{7.93}$$

It can easily be verified that on passing once round the contour L the value of integral (7.93) is zero. Indeed, by taking integral (7.93) round the closed contour of the cross section, and applying the Gauss-Ostrogradsky formula to it, with the first equation of system (7.79) taken into account, we obtain

$$\begin{aligned} \Psi = \int_L \{[(bx_2^2 + ex_2)]\, dx_1 + [-(ax_1^2 + ex_1)]\, dx_2\} = \\ = -2\int_\omega (ax_1 + bx_2 + e)\, d\omega = -2(aS_2 + bS_1 + e\omega) = 0 \end{aligned} \tag{7.94}$$

In a similar way it can also be verified that on passing once round the contour L the value of integral (7.88) is zero. This circumstance and equality (7.94) will be used in what follows.

It is not difficult to check that the values found for the components of the stress tensor σ_{31} and σ_{32} at the end $x_3 = l$ identically satisfy the first two conditions of (7.77). Indeed,

$$\begin{aligned} \int_\omega \sigma_{31} d\omega = \int_\omega \frac{P}{2}\left(\frac{\partial\chi}{\partial x_2} + ax_1^2 + ex_1\right) d\omega = \int_\omega \left\{\frac{P}{2}\left(\frac{\partial\chi}{\partial x_2} + ax_1^2 + ex_1\right) + \right. \\ \left. + x_1\left[\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} - P(ax_1 + bx_2 + e)\right]\right\} d\omega \end{aligned}$$

or

$$\int_{\omega} \sigma_{31} d\omega = \frac{P}{2} \int_{\omega} \left\{ \frac{\partial}{\partial x_1} \left[x_1 \left(\frac{\partial \chi}{\partial x_2} + a x_1^2 + e x_1 \right) \right] + \right.$$
$$\left. + \frac{\partial}{\partial x_2} \left[x_1 \left(- \frac{\partial \chi}{\partial x_1} + b x_2^2 + e x_2 \right) \right] \right\} d\omega - P \left(a I_{22} + b I_{12} + e S_2 \right).$$

By applying the Gauss-Ostrogradsky formula, and using condition (7.88) and the third equation of (7.79), we obtain

$$\int_{\omega} \sigma_{31} d\omega = P.$$

Similarly,

$$\int_{\omega} \sigma_{32} d\omega = 0.$$

The third condition of (7.77) enables one to determine the constant C entering into (7.89). Substituting in this condition the values of σ_{31} and σ_{32} according to (7.85), we obtain

$$M_3 = \int_{\omega} (x_1 \sigma_{32} - x_2 \sigma_{31}) \, d\omega = \frac{P}{2} \int_{\omega} \left[- \left(x_1 \frac{\partial \chi}{\partial x_1} + x_2 \frac{\partial \chi}{\partial x_2} \right) + \right.$$
$$\left. + (b x_2 - a x_1) x_1 x_2 \right] d\omega = - \frac{P}{2} \int_{\omega} \left[\frac{\partial (x_1 \chi)}{\partial x_1} + \frac{\partial (x_2 \chi)}{\partial x_2} \right] d\omega +$$
$$+ P \int_{\omega} \chi d\omega + \frac{P}{2} \int_{\omega} (b x_2 - a x_1) x_1 x_2 d\omega$$

or

$$M_3 = - \frac{P}{2} \int_{L} [x_1 dx_2 - x_2 dx_1] \chi dl + P \int_{\omega} \chi d\omega + \frac{P}{2} \int_{\omega} (b x_2 - a x_1) x_1 x_2 d\omega.$$

We introduce the notation

$$\omega_l = \frac{1}{2} \int_0^l (x_1 dx_2 - x_2 dx_1);$$

then

$$M_3 = - P \int_{L} \chi \frac{d\omega_l}{dl} dl + P \int_{\omega} \chi d\omega + \frac{P}{2} \int_{\omega} (b x_2 - a x_1) x_1 x_2 d\omega.$$

On performing the integration by parts, we find

$$M_3 = P \int_{L} \omega_l \frac{\partial \chi}{\partial l} dl + P \int_{\omega} \chi d\omega + \frac{P}{2} \int_{\omega} (b x_2 - a x_1) x_1 x_2 d\omega. \qquad (7.95)$$

By using formulas (7.88) and (7.89) in (7.95), we have

$$M_3 = P\left\{C\int_{\omega}\Phi d\omega + \int_{\omega}\Psi d\omega + \frac{1}{2}\int_{\omega}(bx_2 - ax_1)\, x_1x_2 d\omega + \right.$$
$$\left. + \int_{L}\left[(bx_2^2 + ex_2)\frac{dx_1}{dl} - (ax_1^2 + ex_1)\frac{dx_2}{dl}\right]\omega_l dl\right\}. \qquad (7.96)$$

It follows from the condition $M_3 = 0$ that

$$C = -\frac{\int_{\omega}\Psi d\omega + \int_{\omega}(bx_2 - ax_1)\, x_1x_2 d\omega + \int_{L} Q dl}{\int_{\omega}\Phi d\omega}, \qquad (7.97)$$

where

$$Q = \left[(bx_2^2 + ex_2)\frac{dx_1}{dl} - (ax_1^2 + ex_1)\frac{dx_2}{dl}\right]\omega_l. \qquad (7.98)$$

If the co-ordinate axes are taken to be the principal centroidal axes, then $S_1 = S_2 = I_{12} = 0$. Consequently, from formulas (7.80) we obtain $a = -1/I_{22}$, $b = e = 0$. In this case the above formulas are appreciably simplified.

67. THE CENTRE OF FLEXURE

By the formulas of Hooke's law (4.50), the components of the strain tensor corresponding to the components of the stress tensor (7.75), (7.76), and (7.85) are

$$\begin{aligned} e_{11} = e_{22} &= -\frac{\nu P}{E}(ax_1 + bx_2 + e)(l - x_3), \quad e_{12} = 0,\\ e_{33} &= \frac{P}{E}(ax_1 + bx_2 + e)(l - x_3),\\ e_{31} &= \frac{(\nu+1)P}{2E}\left(\frac{\partial\chi}{\partial x_2} + ax_1^2 + ex_1\right),\\ e_{32} &= \frac{(1+\nu)P}{2E}\left(-\frac{\partial\chi}{\partial x_1} + bx_2^2 + ex_2\right). \end{aligned} \qquad (7.99)$$

On the basis of formulas (3.27) the angle of rotation of an element of the body about the ox_3 axis is

$$\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

From this formula and formulas (3.26) we find

$$\frac{\partial\omega_3}{\partial x_1} = \frac{\partial e_{12}}{\partial x_1} - \frac{\partial e_{11}}{\partial x_2}, \quad \frac{\partial\omega_3}{\partial x_2} = \frac{\partial e_{22}}{\partial x_1} - \frac{\partial e_{12}}{\partial x_2}, \quad \frac{\partial\omega_3}{\partial x_3} = \frac{\partial e_{23}}{\partial x_1} - \frac{\partial e_{31}}{\partial x_2}.$$

In the same way we can derive similar formulas for partial derivatives of the other angles of rotation, ω_1 and ω_2, with respect to the co-ordinates x_k. The quantity $\frac{\partial \omega_3}{\partial x_3}$ represents the twist of fibres of a prismatic body parallel to the ox_3 axis.

The mean value of the twist for the whole cross section, denoted by τ, is determined by the formula

$$\tau = \frac{1}{\omega} \int\limits_{\omega} \frac{\partial \omega_3}{\partial x_3}\, d\omega = \frac{(1+\nu) P}{E} \left[C - \frac{\nu}{1+\nu} (b x_{1c} - a x_{2c}) \right]. \qquad (7.100)$$

Thus, it appears that under the action of a transverse force applied to a free end of a prismatic body the bending is accompanied by twisting. As seen from formula (7.100), for the prismatic body to undergo only bending, without any twisting, under the action of the indicated force, the constant C must be determined by the formula

$$C = \frac{\nu}{1+\nu} (b x_{1c} - a x_{2c}). \qquad (7.101)$$

Substituting (7.101) in formula (7.96), the twisting moment M_3 is found to be

$$M_3 = P \left\{ \frac{\nu}{1+\nu} (b x_{1c} - a x_{2c}) \int\limits_{\omega} \Phi d\omega + \int\limits_{\omega} \Psi d\omega + \right.$$
$$\left. + \frac{1}{2} \int\limits_{\omega} (b x_2 - a x_1)\, x_1 x_2 d\omega + \int\limits_{L} Q dl \right\}. \qquad (7.102)$$

In order to avoid twisting during the bending of the body, it is necessary, in addition to the force $\boldsymbol{P}$ acting at the point o of the cross section, to apply to this section the twisting moment $\boldsymbol{M}_3$ calculated by formula (7.102). By adding the force $\boldsymbol{P}$ and the twisting moment $\boldsymbol{M}_3$, we obtain a force $\boldsymbol{P}$ equal to the given force, directed parallel to it, and located at a distance x_2^0, which is determined by the formula

$$x_2^0 = -\frac{M_3}{P} = \frac{\nu}{1+\nu} (-b x_{1c} + a x_{2c}) \int\limits_{\omega} \Phi d\omega - \int\limits_{\omega} \Psi d\omega -$$
$$- \frac{1}{2} \int\limits_{\omega} (b x_2 - a x_1)\, x_1 x_2 d\omega - \int\limits_{L} Q\, dl. \qquad (7.103)$$

Suppose, now, that the transverse force $\boldsymbol{P}$ applied at the origin is directed along the ox_2 axis. Reasoning in the same manner, we obtain a force $\boldsymbol{P}$ equal to the given force, directed parallel to it, and located at a distance x_1^0, which is determined by the formula

$$x_1^0 = -\frac{\nu}{1+\nu} (a_* x_{2c} - b_* x_{1c}) \int\limits_{\omega} \Phi d\omega + \int\limits_{\omega} \Psi_* d\omega +$$
$$+ \frac{1}{2} \int\limits_{\omega} (b_* x_2 - a_* x_1)\, x_1 x_2 d\omega + \int\limits_{L} Q\, dl, \qquad (7.104)$$

where

$$Q=\left[(b_*x_2^2+e_*x_2)\frac{dx_1}{dl}-(a_*x_1^2+e_*x_1)\frac{dx_2}{dl}\right]\omega_l.$$

In formula (7.104) the function Ψ_* satisfies the equation

$$\Delta\Psi_*=\frac{2\nu}{1+\nu}(b_*x_1-a_*x_2) \qquad (7.105)$$

and the boundary condition

$$\Psi_*=\int_0^l\left[(b_*x_2^2+e_*x_2)\frac{dx_1}{dl}-(a_*x_1^2+e_*x_1)\frac{dx_2}{dl}\right]dl. \qquad (7.106)$$

In these formulas

$$\begin{gathered}a_*=\frac{I_{12}\omega-S_1S_2}{B}, \quad b_*=\frac{S_2^2-\omega I_{22}}{B},\\ e_*=\frac{I_{22}S_1-I_{12}S_2}{B}=-a_*x_{1c}-b_*x_{2c}.\end{gathered} \qquad (7.107)$$

The point of intersection of the straight lines $x_1=x_1^0$, $x_2=x_2^0$ is called the centre of flexure.

Any transverse force applied to the section at the free end and passing through the centre of flexure produces bending without causing twisting. In order to determine the location of the centre of flexure, it is not at all necessary to solve the problem of bending of a prismatic body, it is sufficient to solve the torsion problem. Following V. V. Novozhilov, let us show that the expressions entering into (7.104) and (7.103) can be calculated with the aid of the function $\Phi(x_1, x_2)$. To prove this, we apply the well-known Green formula for the functions Φ and Ψ; the contour of integration is taken to be the contour of the cross section of the body:

$$\int_\omega(\Phi\Delta\Psi-\Psi\Delta\Phi)\,d\omega=\int_L\left(\Phi\frac{\partial\Psi}{\partial n}-\Psi\frac{\partial\Phi}{\partial n}\right)dl. \qquad (7.108)$$

By using the first equations of (7.90) and (7.91), with condition (7.92), instead of (7.108) we have

$$\frac{2\nu}{1+\nu}\int_\omega(bx_1-ax_2)\,\Phi d\omega+2\int_\omega\Psi d\omega=-\int_L\Psi\frac{\partial\Phi}{\partial n}dl. \qquad (7.109)$$

On the basis of relations (7.13) and (7.10) we have

$$\frac{\partial\varphi}{\partial x_1}=x_2+\frac{\partial\Phi}{\partial x_2}, \qquad \frac{\partial\varphi}{\partial x_2}=-x_1-\frac{\partial\Phi}{\partial x_1}. \qquad (7.110)$$

Taking into account formulas (6.27), we find

$$\frac{\partial\Phi}{\partial n}=\frac{\partial\Phi}{\partial x_1}n_1+\frac{\partial\Phi}{\partial x_2}n_2=-\frac{\partial\varphi}{\partial l}-2\frac{d\omega_l}{dl}. \qquad (7.111)$$

Substituting (7.111) in the right-hand side of (7.109), and integrating by parts, with the use of the second relation of (7.91), we obtain

$$\int_L \Psi \frac{\partial \Phi}{\partial n}\, dl = -\int_L \Psi \left(\frac{\partial \varphi}{\partial l} + 2\frac{d\omega_l}{dl} \right) dl =$$

$$= \int_L (\varphi + 2\omega_l) \frac{\partial \Psi}{\partial l}\, dl = I + 2\int_L Q\, dl, \qquad (7.112)$$

where

$$I = \int_L \{[(bx_2^2 + ex_2)\,\varphi]\, dx_1 + [-(ax_1^2 + ex_1)\,\varphi\,]\, dx_2\}.$$

On the basis of the Gauss-Ostrogradsky formula we have

$$I = -\int_\omega \left\{ \frac{\partial}{\partial x_1} [(ax_1^2 + ex_1)\,\varphi] + \frac{\partial}{\partial x_2} [bx_2^2 + ex_2)\,\varphi\,] \right\} d\omega$$

or

$$I = -2\int_\omega (ax_1 + bx_2 + e)\,\varphi d\omega -$$

$$- \int_\omega \left[(ax_1^2 + ex_1) \frac{\partial \varphi}{\partial x_1} + (bx_2^2 + ex_2) \frac{\partial \varphi}{\partial x_2} \right] d\omega.$$

Substituting the values of $\frac{\partial \varphi}{\partial x_1}$ and $\frac{\partial \varphi}{\partial x_2}$ from (7.110) in this formula, we obtain

$$I = -2\int_\omega (ax_1 + bx_2 + e)\,\varphi d\omega + \int_\omega (bx_2 - ax_1)\, x_1 x_2 d\omega +$$

$$+ \int_\omega \left\{ \frac{\partial\,[(bx_2^2 + ex_2)\,\Phi]}{\partial x_1} - \frac{\partial\,[(ax_1^2 + ex_1)\,\Phi]}{\partial x_2} \right\} d\omega.$$

On integtating the third term by parts, and remembering that $\Phi = 0$ on the contour L, we see that it vanishes. Consequently,

$$I = -2\int_\omega (ax_1 + bx_2 + e)\,\varphi\, d\omega + \int_\omega (bx_2 - ax_1)\, x_1 x_2\, d\omega. \quad (7.113)$$

Inserting (7.112) and (7.113) in (7.109), and then substituting the result in (7.103), we finally obtain a formula for the coordinate of the centre of flexure:

$$x_2^0 = -\int_\omega (ax_1 + bx_2 + e)\,\varphi d\omega + \frac{\nu}{1+\nu}\int_\omega [b\,(x_1 - x_{1c}) -$$

$$- a\,(x_2 - x_{2c})]\,\Phi d\omega. \qquad (7.114)$$

In a similar way, from (7.104) we obtain a formula for the other co-ordinate of the centre of flexure

$$x_1^0 = \int_\omega (a_* x_1 + b_* x_2 + e_*)\,\varphi\, d\omega - \\ -\frac{\nu}{1+\nu}\int_\omega [b_*(x_1 - x_{1c}) - a_*(x_2 - x_{2c})]\,\Phi\, d\omega. \quad (7.115)$$

As seen, the formulas for the determination of the centre of flexure of a prismatic body with a simply connected section involve the functions φ and Φ related only to the solution of the torsion problem. It should be noted that if either of the functions φ, Φ is known, the other is determined by quadrature from (7.110).

In the work* of the author and Bubuteishvili formulas have been derived for the determination of the co-ordinates of the centre of flexure (x_1^0, x_2^0) in the case of a multiply connected region:

$$x_1^0 = -\int_\omega (a_* x_1 + b_* x_2 + e_*)\,\mathrm{Im}\,F(z)\, d\omega - \frac{\nu}{\nu+1}\Big\{\int_\omega [b_*(x_1 - x_{1c}) - \\ - a_*(x_2 - x_{2c})]\,(\mathrm{Re}\,F(z) - \frac{1}{2}(x_1^2 + x_2^2))\, d\omega - \\ - [b_*(x_{1c}^0 - x_{1c}) - a_*(x_{2c}^0 - x_{2c})]\,C_0\omega_0 + \sum_{k=1}^{n}[b_*(x_{1c}^k - x_{1c}) - \\ - a_*(x_{2c}^k - x_{2c})]\,C_k\omega_k\Big\},$$

$$x_2^0 = \int_\omega (a x_1 + b x_2 + e)\,\mathrm{Im}\,F(z)\, d\omega + \frac{\nu}{\nu+1}\Big\{\int_\omega [b(x_1 - x_{1c}) - \\ - a(x_2 - x_{2c})\Big(\mathrm{Re}\,F(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big)\, d\omega - [b(x_{1c}^0 - x_{1c}) - \\ - a(x_{2c}^0 - x_{2c})]\,C_0\omega_0 + \sum_{k=1}^{n}[b(x_{1c}^k - x_{1c}) - a(x_{2c}^k - x_{2c})]\,C_k\omega_k\Big\},$$

where $F(z)$ is the complex torsion function; x_{2c}^k, x_{1c}^k are the co-ordinates of the centroid of the area enclosed by the contour L_k; x_{1c}, x_{2c} are the co-ordinates of the centroid of the cross-sectional area; ω_k is the area enclosed by the contour L_k; C_k are some constants introduced in (7.20). The constants (a, b, e), (a_*, b_*, e_*) are determined, respectively, by formulas (7.80) and (7.107).

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* Yu. A. Amenzade, O. L. Bubuteishvili, *The Centre of Flexure of a Cantilever with a Multiply Connected Cross Section*, Doklady Akad. Nauk Azerb. SSR, **29** (10), 3-6 (1973) (in Russian).

Consider the problem of determining the centre of flexure when the section of a cantilever represents a region bounded externally by a circumference L_0 of radius R, and internally by a circumference L_1 of radius r (Fig. 34). The approximate expression for the complex torsion function $F(z)$ in this problem is determined by formula (7.62).

For the adopted co-ordinate system, $e = a = 0$, $b = 1.288\ R^{-4}$. For the relative dimensions indicated in the problem (e), $x_1^0 = -0.34789\ R$, $x_2^0 = 0$.

68. BENDING OF A PRISMATIC BODY OF ELLIPTICAL CROSS SECTION

Let the planes x_1ox_3 and x_2ox_3 be planes of symmetry of a prismatic body, and let the load acting on its end be statically equivalent to a force $\boldsymbol{P}$, which is directed along the x_1 axis and applied at the centre of the end. In these conditions the body will obviously act in bending, without any twisting.

On the basis of formulas (7.80) we have

$$a = -\frac{1}{I_{22}}, \qquad b = e = 0.$$

Inserting these relations in (7.91), we have

$$\Delta\Psi = \frac{2\nu}{(1+\nu)\,I_{22}}\,x_2, \tag{7.116}$$

$$\frac{\partial\Psi}{\partial l} = \frac{1}{I_{22}}\,x_1^2\,\frac{\partial x_2}{dl} \quad \text{on } L. \tag{7.117}$$

Following the procedure of Timoshenko, we introduce, instead of the flexure function Ψ, a new function:

$$Q(x_1, x_2) = \Psi(x_1, x_2) + f(x_2), \tag{7.118}$$

where $f(x_2)$ is an arbitrary function of x_2 only.

Substituting (7.118) in (7.117), we obtain the boundary conditions on L

$$\frac{\partial Q}{\partial l} = \left(\frac{1}{I_{22}}\,x_1^2 + f'\right)\frac{dx_2}{dl}. \tag{7.119}$$

In a particular case the function $f(x_2)$ may be chosen so as to make the bracketed expression zero; the boundary condition on L then simplifies to

$$\frac{\partial Q}{\partial l} = 0. \tag{7.120}$$

Since the cross section is simply connected, (7.120) on L may be written as

$$Q = 0. \tag{7.121}$$

For example, in the case when the cross section is an ellipse equality (7.120) can be satisfied by setting

$$f(x_2)=\frac{a_1^2}{I_{22}}x_2\left(\frac{1}{3}\frac{x_2^2}{b_1^2}-1\right). \tag{7.122}$$

Here a_1 and b_1 are the semiaxes of the ellipse.

Consider the problem of the bending of an elliptical cylinder. For the given problem, with (7.122), the function Q becomes

$$Q(x_1, x_2)=\Psi(x_1, x_2)+\frac{a_1^2}{I_{22}}x_2\left(\frac{1}{3}\frac{x_2^2}{b_1^2}-1\right). \tag{7.123}$$

Inserting (7.123) in (7.116), we obtain an equation of the form

$$\Delta Q = nx_2, \tag{7.124}$$

where

$$n=\frac{2}{I_{22}}\left(\frac{\nu}{1+\nu}+\frac{a_1^2}{b_1^2}\right). \tag{7.125}$$

The solution of Eq. (7.124) is sought in the form

$$Q=K\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{b_1^2}-1\right)x_2. \tag{7.126}$$

This solution satisfies the boundary condition (7.121).

Substituting (7.126) in Eq. (7.124), we have

$$2K\left(\frac{1}{a_1^2}+\frac{3}{b_1^2}\right)=n,$$

from which

$$K=\frac{1}{I_{22}}\frac{a_1^2b_1^2}{b_1^2+3a_1^2}\left(\frac{\nu}{1+\nu}+\frac{a_1^2}{b_1^2}\right). \tag{7.127}$$

Knowing the function Q, we can determine the function Ψ from (7.123):

$$\Psi=K\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{b_1^2}-1\right)x_2-\frac{a_1^2}{I_{22}}x_2\left(\frac{1}{3}\frac{x_2^2}{b_1^2}-1\right). \tag{7.128}$$

By using formulas (7.85), and noting that $\chi = \Psi$ for the given problem, we have

$$\sigma_{13}=\frac{P}{2}\left\{K\left(\frac{x_1^2}{a_1^2}+\frac{3x_2^2}{b_1^2}-1\right)-\frac{a_1^2}{I_{22}}\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{b_1^2}-1\right)\right\}, \tag{7.129}$$

$$\sigma_{23}=-PK\frac{x_1x_2}{a_1^2};$$

from formula (7.76),

$$\sigma_{33} = -\frac{P}{I_{22}} x_1 (l - x_3). \tag{7.130}$$

The last formula for the normal stress is completely coincident with the formula of the elementary theory of bending, but this cannot be said about the formulas for the tangential stresses σ_{13} and σ_{23}. According to the elementary theory of bending, in the problem considered $\sigma_{23} = 0$ and σ_{13} depends only on x_1.

On the axis $x_1 = 0$ we have

$$\sigma_{13} = \frac{Pa_1^2}{2I_{22}(1+\nu)(b_1^2+3a_1^2)} \{b_1^2 + 2(1+\nu)a_1^2 - x_2^2(1-2\nu)\}, \tag{7.131}$$

$$\sigma_{23} = 0,$$

whereas, by Zhuravskii's formula, on this axis

$$\sigma_{13} = \frac{Pa_1^2}{3I_{22}}. \tag{7.132}$$

If the material is incompressible, i.e., $\nu = 0.5$, formulas (7.131) and (7.132) are identical.

In the present chapter we have considered the theory of torsion and bending of prismatic bodies, which is of great importance in engineering. Here we leave detailed discussion of a large number of special problems examined by many authors.

CHAPTER VIII

General theorems of the theory of elasticity. Variational methods

In the present chapter we shall consider some well-known principles of the theory of elasticity, which are of great importance in the development of a variety of very effective methods for the numerical solution of boundary value problems in elasticity. In Chap. IV we have become acquainted with one of the general theorems of the theory of elasticity, namely Clapeyron's theorem.

69. BETTI'S RECIPROCAL THEOREM

Let σ'_{kr}, u'_k, e'_{kr} and σ''_{kr}, u''_k, e''_{kr} denote, respectively, the components of the stress tensor, the displacement vector, and the strain tensor, which are produced in an elastic body by external forces $\rho \boldsymbol{F}'$, $\boldsymbol{T}'_n$ and $\rho \boldsymbol{F}''$, $\boldsymbol{T}''_n$.

The work done by the forces $\rho \boldsymbol{F}'$, $\boldsymbol{T}'_n$, including the inertia forces $-\rho \frac{\partial^2 u'_k}{\partial t^2}$, during the displacements u''_k is

$$A_{12} = \int\limits_{\tau} \left[\left(F'_k - \frac{\partial^2 u'_k}{\partial t^2} \right) u''_k \right] \rho \, d\tau + \int\limits_{\omega} (T'_{nk} u''_k) \, d\omega.$$

Inserting $T'_{nk} = \sigma'_{kr} n_r$ in the last expression, and remembering that the stress tensor is symmetrical, after transforming the surface integral into a volume one, we have

$$A_{12} = \int\limits_{\tau} \left[\frac{\partial \sigma'_{kr}}{\partial x_r} + \rho \left(F'_k - \frac{\partial^2 u'_k}{\partial t^2} \right) \right] u''_k \, d\tau + \int\limits_{\tau} \sigma'_{kr} e''_{kr} \, d\tau.$$

By (2.24),

$$A_{12} = \int\limits_{\tau} \sigma'_{kr} e''_{kr} \, d\tau. \tag{8.1}$$

The work done by the forces $\rho \boldsymbol{F}''$, $\boldsymbol{T}''_n$, $-\rho \frac{\partial^2 u''}{\partial t^2}$ during the displacements u'_k is

$$A_{21} = \int\limits_{\tau} \left[\left(F''_k - \frac{\partial^2 u''_k}{\partial t^2} \right) u'_k \right] \rho \, d\tau + \int\limits_{\omega} (T''_{nk} u'_k) \, d\omega.$$

On carrying out transformations for A_{21} similar to those for A_{12}, we obtain

$$A_{21} = \int_{\tau} \sigma''_{kr} e'_{kr}\, d\tau. \qquad (8.2)$$

By Betti's identity (4.61), from (8.1) and (8.2) we obtain

$$A_{21} = A_{12}. \qquad (8.3)$$

This is Betti's reciprocal theorem. It states that the work done by the first system of external forces during the displacements of an elastic body produced by the second system of external forces is equal to the work done by the second system of external forces during the displacements of the same body produced by the first force system.

70. PRINCIPLE OF MINIMUM POTENTIAL ENERGY

Let the actual displacement vector be denoted by $\boldsymbol{u}$, and the corresponding stress tensor by σ_{mk}. This stress tensor satisfies the differential equations of equilibrium

$$\frac{\partial \sigma_{mk}}{\partial x_m} + \rho F_k = 0 \qquad (8.4)$$

and the surface conditions

$$T_{nm} = \sigma_{mk} n_k. \qquad (8.5)$$

If the displacement vector is given a variation $\delta \boldsymbol{u}$, then from the equality $\boldsymbol{u}^* = \boldsymbol{u} + \delta \boldsymbol{u}$ and formulas (3.26) we have

$$e_{mk} + \delta e_{mk} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m}\right) + \frac{1}{2}\left(\delta\,\frac{\partial u_m}{\partial x_k} + \delta\,\frac{\partial u_k}{\partial x_m}\right). \qquad (8.6)$$

From this we find the change in the strain tensor

$$\delta e_{mk} = \frac{1}{2}\left(\delta \frac{\partial u_m}{\partial x_k} + \delta \frac{\partial u_k}{\partial x_m}\right). \qquad (8.7)$$

Denoting the deformation work per unit volume for the varied state of equilibrium by $A\,(e_{mk} + \delta e_{mk})$, and expanding its expression in a Taylor series, we obtain

$$A\,(e_{mk} + \delta e_{mk}) = A\,(e_{mk}) + \left(\frac{\partial A}{\partial e_{11}}\,\delta e_{11} + \ldots + \frac{\partial A}{\partial e_{31}}\,\delta e_{31}\right) +$$
$$+ \frac{1}{2}\left(\frac{\partial^2 A}{\partial e_{11}^2}\,\delta e_{11}^2 + \ldots + 2\,\frac{\partial^2 A}{\partial e_{11}\,\partial e_{22}}\,\delta e_{11}\delta e_{22} + \ldots + \frac{\partial^2 A}{\partial e_{31}^2}\,\delta e_{31}^2\right). \qquad (8.8)$$

Here $A\,(e_{mk})$ is the value of the deformation work per unit volume in the actual state of equilibrium. Taking into account that

$$\frac{\partial^2 A}{\partial e_{kk}\,\partial e_{rr}} = \frac{\partial \sigma_{rr}}{\partial e_{kk}}, \quad \frac{\partial^2 A}{\partial e_{kr}^2} = \frac{\partial\,(2\sigma_{kr})}{\partial e_{kr}} \quad (k \neq r),$$

and using Hooke's law, the last term in (8.8) may be put into the form

$$\frac{1}{2}[\lambda\delta\theta^2+2\mu\delta e_{mk}\delta e_{km}]. \tag{8.9}$$

Expression (8.9), as is known from (4.36), represents the deformation work per unit volume corresponding to the variation of the displacement vector $\delta \boldsymbol{u}$, and is always positive definite.

By using (4.20) and (8.7), the second term in (8.8) is transformed into

$$\sigma_{11}\delta\frac{\partial u_1}{\partial x_1}+\ldots+\sigma_{31}\delta\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)=\sigma_{mk}\frac{\partial\delta u_m}{\partial x_k}. \tag{8.10}$$

Let the stress vectors on the co-ordinate planes be denoted by $\boldsymbol{T}_m$; instead of (8.10) we then have

$$\sigma_{mk}\frac{\partial\delta u_m}{\partial x_k}=\boldsymbol{T}_m\cdot\nabla\delta u_m, \tag{8.11}$$

where ∇ is the Hamiltonian operator, $\nabla = \boldsymbol{i}_k\frac{\partial}{\partial x_k}$.

From (8.8), with (8.9) and (8.11), we find

$$\delta R=\int_\tau \boldsymbol{T}_m\cdot\nabla\delta u_m\,d\tau+\int_\tau\frac{1}{2}(\lambda\delta\theta^2+2\mu\delta e_{mk}\delta e_{km})\,d\tau; \tag{8.12}$$

this represents the increment of the work of deformation. It can easily be found by direct checking that

$$\operatorname{div}\delta u_m\boldsymbol{T}_m=\delta u_m\operatorname{div}\boldsymbol{T}_m+\boldsymbol{T}_m\cdot\nabla\delta u_m \tag{8.13}$$

(here the index m is not summed), from which

$$\int_\tau \boldsymbol{T}_m\cdot\nabla\delta u_m\,d\tau=\int_\tau\operatorname{div}\delta u_m\boldsymbol{T}_m\,d\tau-\int_\tau\delta u_m\operatorname{div}\boldsymbol{T}_m\,d\tau. \tag{8.14}$$

With the use of the Gauss-Ostrogradsky formula we obtain

$$\int_\tau\operatorname{div}\delta u_m\boldsymbol{T}_m\,d\tau=\int_\omega\delta u_m\sigma_{mk}n_k\,d\omega. \tag{8.15}$$

The integrand on the right-hand side of the last equality is summed with respect to the index k; n_k are the cosines of the angles between the normal $\boldsymbol{n}$ and the co-ordinate axes x_k.

From (8.5) and (8.15) we find

$$\int_\tau\operatorname{div}\delta u_m\boldsymbol{T}_m\,d\tau=\int_\omega T_{nm}\delta u_m\,d\omega.$$

Also, by the equilibrium equation (2.27),

$$\int_\tau\delta u_m\operatorname{div}\boldsymbol{T}_m\,d\tau=\int_\tau-\rho F_m\delta u_m\,d\tau.$$

Consequently, from (8.12), with (8.14), we have

$$\delta R = \int_\tau \rho F_m \delta u_m \, d\tau + \int_\omega T_{nm} \delta u_m \, d\omega + \int_\tau \frac{1}{2} (\lambda \delta\theta^2 + 2\mu \delta e_{mk} \delta e_{km}) \, d\tau. \tag{8.16}$$

Let ω_u be the sum of parts of the surface over which the displacement vector assumes given values, and let ω_T be the remaining part of the surface over which the forces T_{nm} are given. Taking into account that $\delta u_m = 0$ over ω_u, where the surface forces are not known, and that the surface forces T_{nk} over ω_T, as well as the body forces, are not subject to the variation, from (8.16) we find

$$\delta \left(R - \int_\tau \rho F_m u_m \, d\tau - \int_{\omega_T} T_{nm} u_m \, d\omega \right) = \frac{1}{2} \int_\tau [\lambda \delta\theta^2 + 2\mu \delta e_{mk} \delta e_{km}] \, d\tau \tag{8.17}$$

or

$$\delta \Pi = \varepsilon,$$

where

$$\Pi = R - \int_\tau \rho F_m u_m \, d\tau - \int_{\omega_T} T_{nm} u_m \, d\omega,$$

$$\varepsilon = \frac{1}{2} [\lambda \delta\theta^2 + 2\mu \delta e_{mk} \delta e_{km}] \, d\tau.$$

Here R is the work of deformation corresponding to the actual displacements; $\int_\tau \rho F_m u_m d\tau$ is the work done by the volume forces during the actual displacements; $\int_{\omega_T} T_{nm} u_m d\omega$ is the work done by the given surface forces over ω_T during the displacements u_m; Π is the potential energy of the body; ε is a positive definite quantity.

Equality (8.17) enables one to formulate the following theorem: the potential energy of an elastic body, considered as a functional of an arbitrary system of displacements satisfying the kinematic boundary conditions, takes a minimum value for the system of displacements actually realized in the elastic body.

71 PRINCIPLE OF MINIMUM COMPLEMENTARY WORK—CASTIGLIANO'S PRINCIPLE

Consider, now, equilibrium in which the displacements u_k and the corresponding stress tensor σ_{mk} are given. In passing from the actual state of stress σ_{mk} to a neighbouring state of stress $\sigma_{mk} +$

$+ \delta\sigma_{mk}$, the change in the deformation work per unit volume is

$$\delta A = A(\sigma_{mk} + \delta\sigma_{mk}) - A(\sigma_{mk}).$$

The varied stress tensor $\sigma_{mk} + \delta\sigma_{mk}$, just as the actual stress tensor σ_{mk}, must satisfy the differential equations of equilibrium, i.e.,

$$\frac{\partial\sigma_{mk}}{\partial x_m} + \rho F_k = 0, \quad \frac{\partial(\sigma_{mk} + \delta\sigma_{mk})}{\partial x_m} + \rho F_k = 0,$$

from which it follows that

$$\frac{\partial\delta\sigma_{mk}}{\partial x_m} = 0. \tag{8.18}$$

By expanding the expression for the deformation work per unit volume $A(\sigma_{mk} + \delta\sigma_{mk})$ in a Taylor series, we obtain

$$\delta A = A(\sigma_{mk} + \delta\sigma_{mk}) - A(\sigma_{mk}) = \left(\frac{\partial A}{\partial\sigma_{11}}\delta\sigma_{11} + \ldots + \frac{\partial A}{\partial\sigma_{31}}\delta\sigma_{31}\right) + \\ + \frac{1}{2}\left(\frac{\partial^2 A}{\partial\sigma_{11}^2}\delta\sigma_{11}^2 + \ldots + 2\frac{\partial^2 A}{\partial\sigma_{11}\,\partial\sigma_{22}}\delta\sigma_{11}\delta\sigma_{22} + \ldots + \frac{\partial^2 A}{\partial\sigma_{31}^2}\delta\sigma_{31}^2\right). \tag{8.19}$$

The second term on the right-hand side of the second equality of (8.19), as in the case of (8.9), represents the deformation work per unit volume corresponding to the variation of the stress tensor $\delta\sigma_{mk}$, and is always positive definite.

By using formulas (4.27), we transform the first term on the right-hand side of the second equality of (8.19) into

$$\frac{\partial u_m}{\partial x_k}\delta\sigma_{mk}$$

or

$$\delta\boldsymbol{T}_m \cdot \nabla u_m.$$

Also, similarly to (8.13), we have

$$\delta\boldsymbol{T}_m \cdot \nabla u_m = \operatorname{div}(u_m \delta\boldsymbol{T}_m) - u_m \operatorname{div}\delta\boldsymbol{T}_m. \tag{8.20}$$

With (8.18) taken into account in (8.20), we obtain

$$\delta\boldsymbol{T}_m \cdot \nabla u_m = \operatorname{div}(u_m \delta\boldsymbol{T}_m).$$

Consequently, from (8.19) we have

$$\delta R = \int_\tau \operatorname{div}(u_m \delta\boldsymbol{T}_m)\,d\tau + \\ + \int_\tau \frac{1}{2}\left(\frac{\partial^2 A}{\partial\sigma_{11}^2}\delta\sigma_{11}^2 + \ldots + 2\frac{\partial^2 A}{\partial\sigma_{11}\,\partial\sigma_{22}}\delta\sigma_{11}\delta\sigma_{22} + \ldots + \frac{\partial^2 A}{\partial\sigma_{31}^2}\delta\sigma_{31}^2\right)d\tau.$$

If (4.27) and the formulas of Hooke's law (4.50) are taken into account in the integrand of the second integral, we obtain the work of deformation expressed in terms of the variations of the stress tensor $\delta\sigma_{kr}$.

By applying the Gauss-Ostrogradsky formula to the first term, and denoting the second integral by ε, we have

$$\delta R = \int_{\omega} u_m \delta\sigma_{mk} n_k \, d\omega + \varepsilon = \int_{\omega} u_m \delta T_{nm} \, d\omega + \varepsilon \qquad (8.21)$$

We impose on the varied stress tensor $\sigma_{mk} + \delta\sigma_{mk}$ the condition that it should be balanced by the given surface forces. Then $\delta T_n = 0$ on parts of the surface where the forces are prescribed. Hence, (8.21) becomes

$$\delta R = \int_{\omega_u} u_m \delta T_{nm} \, d\omega + \varepsilon.$$

Here ω_u is the sum of parts of the surface over which the displacements are prescribed. Noting that the displacements u_m on ω_u are not subject to the variation, the last formula is rearranged in the form

$$\delta R^* = \varepsilon, \qquad (8.22)$$

where

$$R^* = R - \int_{\omega_u} u_m T_{nm} \, d\omega.$$

Here $\int_{\omega_u} u_m T_{nm} \, d\omega$ is the work done by the surface forces during the given displacements on ω_u; R^* is called the complementary work.

Remembering that ε is always a positive definite quantity, we come to the conclusion that R^* assumes a minimum value.

Equality (8.22) enables one to formulate the following theorem: the complementary work of an elastic body, considered as a functional of an arbitrary stress system satisfying the equilibrium equations within the body and on its surface, takes a minimum value for the stress system actually realized in the elastic body.

72. RAYLEIGH-RITZ METHOD

The solution of an elasticity problem often involves great mathematical difficulties. In these cases recourse is made to the principles of minimum potential or complementary energy. The application of these principles consists in finding functions satisfying the boundary conditions of the problem and in minimizing the potential energy Π or the complementary energy R^*.

One of the most efficient variational methods is the Rayleigh-Ritz method. In this method the solution is represented in the form of an expression satisfying the boundary conditions and containing unknown coefficients c_k, where $k = 1, 2, 3, 4, \ldots$. Next, the value of the potential or complementary energy is calculated. The expressions thus obtained are functions of the coefficients c_k. These coefficients for the actual equilibrium state can be determined from the conditions for minimizing the potential or complementary energy, i.e.,

$$\frac{\partial \Pi}{\partial c_k}=0 \text{ or } \frac{\partial R^*}{\partial c_k}=0 \quad (k=1, 2, 3, 4 \ldots \infty).$$

If $k = 1, 2, 3, \ldots, n$, then $\frac{\partial \Pi}{\partial c_k}$ or $\frac{\partial R^*}{\partial c_k} = 0$ lead to a system of n linear equations in the coefficients c_k. Substituting the values of these coefficients in the above expression, we obtain an approximate solution of the problem. It should be noted that the solution of the problem obtained in this way is exact if the assumed expression includes a complete sequence of functions, i.e., a sequence of measurable functions of class C, where an arbitrary function from this class can be approximated with the required accuracy by means of a linear combination of a finite number of these functions. In most cases, however, it is possible to take into account only a finite number of coefficients c_k.

As an example let us consider the unconstrained torsion of prismatic bars. Noting that in torsion $e_{11} = e_{22} = e_{33} = \theta = e_{12} = 0$, $e_{31} = \sigma_{31}/2\mu$, and $e_{32} = \sigma_{32}/2\mu$, from formula (4.36) we obtain the amount of strain energy stored in a bar of length a:

$$R=\frac{a}{2\mu}\int\limits_{\omega}(\sigma_{31}^2+\sigma_{32}^2)\,dx_1\,dx_2,$$

where ω is the cross-sectional area of the bar.

By (7.14), the last formula becomes

$$R=\frac{\mu\tau^2 a}{2}\int\limits_{\omega}\left[\left(\frac{\partial\Phi}{\partial x_1}\right)^2+\left(\frac{\partial\Phi}{\partial x_2}\right)^2\right]dx_1\,dx_2.$$

The given surface forces on the lateral surface of the bar are zero; hence, the work on this surface vanishes, and at both ends the work is

$$\int\limits_{\omega}[u_1T_{n1}+u_2T_{n2}]\,dx_1\,dx_2\,|_{x_3=0}+\int\limits_{\omega}[u_1T_{n1}+u_2T_{n2}]\,dx_1\,dx_2\,|_{x_3=a}. \quad (8.23)$$

Here, by formulas (7.1), we have $u_1 = u_2 = 0$ when $x_3 = 0$; when $x_3 = a$, we have $u_1 = -\tau x_2 a$, $u_2 = \tau x_1 a$, and by formulas (2.22), with $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12}=0$, we have $T_{n1} = \sigma_{31}$, $T_{n2} = \sigma_{32}$

when $x_3 = a$. Instead of (8.23) we then have

$$\mu\tau^2 a \int\limits_{\omega} \left(-x_2 \frac{\partial \Phi}{\partial x_2} - x_1 \frac{\partial \Phi}{\partial x_1}\right) dx_1\, dx_2.$$

The last expression is rearranged in the form

$$2\mu\tau^2 a \int\limits_{\omega} \Phi\, dx_1\, dx_2 - \mu\tau^2 a \int\limits_{\omega} \left(\frac{\partial (x_1\Phi)}{\partial x_1} + \frac{\partial (x_2\Phi)}{\partial x_2}\right) d\omega,$$

from which, with the aid of the Gauss-Ostrogradsky formula, we obtain

$$2\mu\tau^2 a \int\limits_{\omega} \Phi\, dx_1\, dx_2 - \mu\tau^2 a \int\limits_{l} \Phi\, (x_1\, dx_2 + x_2\, dx_1),$$

where l is the contour of the region occupied by the cross section of the bar.

Thus, the complementary energy is, by definition,

$$R^* = \frac{\mu\tau^2 a}{2} \int\limits_{\omega} \left[\left(\frac{\partial \Phi}{\partial x_1}\right)^2 + \left(\frac{\partial \Phi}{\partial x_2}\right)^2 - 4\Phi\right] dx_1\, dx_2 + \\ + \mu\tau^2 a \int\limits_{l} \Phi\, (x_1\, dx_2 + x_2\, dx_1).$$

By (7.16), $\Phi =$ constant on the contour l; on the other hand, this constant may be taken equal to zero; we then have, finally,

$$R^* = \frac{\mu\tau^2 a}{2} \int\limits_{\omega} \left[\left(\frac{\partial \Phi}{\partial x_1}\right)^2 + \left(\frac{\partial \Phi}{\partial x^2}\right)^2 - 4\Phi\right] dx_1\, dx_2. \tag{8.24}$$

Take a bar of rectangular section of sides $2b$ and $2c$. Noting that Prandtl's stress function Φ on the sides $x_1 = \pm b$ and $x_2 = \pm c$ must be zero and symmetrical in x_1 and x_2, we include in its expression only terms with even powers of x_1 and x_2, i.e.,

$$\Phi = (x_1^2 - b^2)(x_2^2 - c^2)(c_1 + c_2 x_1^2 + c_3 x_2^2 + c_4 x_1^2 x_2^2 + \ldots).$$

As a first approximation we take the expression

$$\Phi = c_1 (x_1^2 - b^2)(x_2^2 - c^2). \tag{8.25}$$

Substituting (8.25) in formula (8.24), we find

$$R^* = \frac{\mu\tau^2 a}{2} \int\limits_{-b}^{b} \int\limits_{-c}^{c} \{4c_1^2 [x_1^2 (x_2^2 - c^2)^2 + (x_1^2 - b^2)^2 x_2^2] - \\ - 4c_1 (x_1^2 - b^2)(x_2^2 - c^2)\}\, dx_1\, dx_2,$$

from which

$$R^* = \frac{\mu\tau^2 a}{2} \frac{64}{45} [2c_1^2 b^3 c^3 (b^2 + c^2) - 5c_1 b^3 c^3].$$

Since the complementary energy for the actual equilibrium state must assume a minimum value, it follows that

$$\frac{\partial R^*}{\partial c_1} = 0,$$

from which

$$c_1 = \frac{5}{4(b^2 + c^2)}. \tag{8.26}$$

By means of formula (7.19) we determine the twisting moment

$$M_t = 2\mu\tau \int_\omega \Phi \, dx_1 \, dx_2 = 2c_1\mu\tau \int_{-b}^{b} \int_{-c}^{c} (x_1^2 - b^2)(x_2^2 - c^2) \, dx_1 \, dx_2 =$$

$$= \frac{32}{9} b^3 c^3 c_1 \mu\tau. \tag{8.27}$$

The maximum shearing stress $T_{\max}$ occurring at the middle of the longer side ($b > c$) is

$$T_{\max} = \mu\tau \left. \frac{\partial \Phi}{\partial x_2} \right|_{\substack{x_2 = -c \\ x_1 = 0}} = 2b^2 c c_1 \mu\tau. \tag{8.28}$$

Substituting the value of c_1 from (8.27) in formula (8.28), we find

$$T_{\max} = \frac{9}{16} \frac{c}{b} \frac{M_t}{c^3}.$$

By formulas (8.26) and (8.27), the torsional rigidity of the bar is

$$D = \frac{M_t}{\tau} = \frac{40}{9} \mu \frac{\left(\frac{c}{b}\right)^3}{1 + \left(\frac{c}{b}\right)^2} b^4.$$

In the case of a bar of square section the approximate solution gives the value of the rigidity $D = 2.222\mu b^4$ and $T_{\max} = 0.563 M_t/b^3$, whereas the exact values are $D = 2.250\mu b^4$ and $T_{\max} = 0.600 M_t/b^3$; the errors are —1.2 and —6.2 per cent, respectively.

We now take the stress function Φ in the form

$$\Phi = (x_1^2 - b^2)(x_2^2 - c^2)(c_1 + c_2 x_1^2 + c_3 x_2^2).$$

Then

$$R^* = \frac{\mu\tau^2 a}{2} \frac{64}{4725} b^3 c^3 [210(b^2 + c^2) c_1^2 + b^4 (66c^2 + 10b^2) c_2^2 +$$

$$+ c^4 (66b^2 + 10c^2) c_3^2 + b^2 (84c^2 + 60b^2) c_1 c_2 + c^2 (84b^2 + 60c^2) c_1 c_3 +$$

$$+ 12b^2c^2 (b^2 + c^2) c_2 c_3 - 525c_1 - 105b^2 c_1 - 105c^2 c_3]$$

From the condition for minimizing the complementary energy

$$\frac{\partial R^*}{\partial c_1}=0, \quad \frac{\partial R^*}{\partial c_2}=0, \quad \frac{\partial R^*}{\partial c_3}=0$$

we arrive at a linear system of three equations:

$$\begin{aligned}
140\,(b^2+c^2)\,c_1+b^2\,(28c^2+20b^2)\,c_2+c^2\,(28b^2+20c^2)\,c_3 &= 175,\\
(84c^2+60b^2)\,c_1+b^2\,(132c^2+20b^2)\,c_2+12c^2\,(b^2+c^2)\,c_3 &= 105,\\
(84b^2+60c^2)\,c_1+12b^2\,(b^2+c^2)\,c_2+c^2\,(132b^2+20c^2)\,c_3 &= 105.
\end{aligned}$$

For the case $c/b = 1$ we find

$$c_1=\frac{1295}{2216b^2}, \quad c_2=c_3=\frac{525}{4432b^2}.$$

Then

$$M_t=2\mu\tau\int_{S_2}\Phi\,dx_1\,dx_2=2.246b^4\mu\tau,$$

$$T_{\max}=\mu\tau\left.\frac{\partial\Phi}{\partial x_1}\right|_{\substack{x_1=b\\ x_2=0}}=0.626\frac{M_t}{b^3}.$$

The errors are now —0.18 and +4.3 per cent, respectively.

It appears from the foregoing numerical examples that as the number of unknown coefficients is increased, the accuracy of the solution is improved. If the exact solution of the problem is not known, the only way of getting an approximate idea of the accuracy of the solution is to increase successively the number of unknown coefficients and compare the final results. If the results converge rapidly, the approximation may be regarded as good.

73. REISSNER'S VARIATIONAL PRINCIPLE

In Secs. 71 and 72 we have presented two well-known variational principles in elasticity: the principle of minimum potential energy, also called the principle of virtual displacements, and the principle of minimum complementary work referred to as Castigliano's principle.

E. Reissner proposed a variational principle that also furnishes the means of finding approximate solutions of elasticity problems. In this principle both the stress tensor and the displacements are varied independently of each other.

Reissner's variational principle is that the variational equation

$$\delta I = 0, \tag{8.29}$$

where I is Reissner's functional,

$$I=\int_\tau\left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_r}+\frac{\partial u_r}{\partial x_k}\right)\sigma_{kr}-A\right]d\tau-\int_\omega \overline{T}_{nk}u_k\,d\omega, \tag{8.30}$$

is equivalent to a system of six relations between the components of the stress tensor and the strain tensor

$$\frac{1}{2}\left(\frac{\partial u_k}{\partial x_r}+\frac{\partial u_r}{\partial x_k}\right)=\frac{\partial A}{\partial \sigma_{kr}^*} \tag{8.31}$$

$$(\sigma_{kr}^* = \sigma_{kr} \text{ when } k = r, \quad \sigma_{kr}^* = 2\sigma_{kr} \text{ when } k \neq r),$$

to three equilibrium equations (for simplicity, body forces are disregarded)

$$\frac{\partial \sigma_{kr}}{\partial x_k}=0, \tag{8.32}$$

and the boundary conditions

$$T_{nk}=\overline{T}_{nk} \text{ on } \omega_T, \tag{8.33}$$

$$u_k=\overline{u}_k \text{ on } \omega_u. \tag{8.34}$$

Here ω_T is the sum of parts of the surface over which the forces $\overline{T}_{nk}$ are given, and ω_u is the remaining part of the surface over which the displacements $\bar{u}_k$ are given.

To prove this principle, we use the well-known relations

$$\delta\left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_r}+\frac{\partial u_r}{\partial x_k}\right)\right]=\frac{1}{2}\left(\frac{\partial \delta u_k}{\partial x_r}+\frac{\partial \delta u_r}{\partial x_k}\right).$$

From (8.29) and (8.30) we then obtain

$$\int_\tau \left[\frac{1}{2}\left(\frac{\partial \delta u_k}{\partial x_r}+\frac{\partial \delta u_r}{\partial x_k}\right)\sigma_{kr}+\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial u_k}{\partial x_r}+\frac{\partial u_r}{\partial x_k}\right)\delta\sigma_{kr}-\frac{\partial A}{\partial \sigma_{kr}^*}\delta\sigma_{kr}\right]d\tau-\int_{\omega_T}\overline{T}_{nk}\delta u_k\,d\omega=0.$$

By using the Gauss-Ostrogradsky formula, we have

$$\int_\tau \frac{1}{2}\left(\frac{\partial \delta u_k}{\partial x_r}+\frac{\partial \delta u_r}{\partial x_k}\right)\sigma_{kr}\,d\tau=\int_\omega \sigma_{kr}n_r\delta u_k\,d\omega-\int_\tau \frac{\partial \sigma_{kr}}{\partial x_r}\delta u_k\,d\tau.$$

Noting that $\sigma_{kr}n_r=T_{nk}$ and $\omega=\omega_T+\omega_u$, we obtain

$$\int_\tau \frac{1}{2}\left(\frac{\partial \delta u_k}{\partial x_r}+\frac{\partial \delta u_r}{\partial x_k}\right)\sigma_{kr}\,d\tau=\int_{\omega_T} T_{nk}\delta u_k\,d\omega+\int_{\omega_u} T_{nk}\delta u_k\,d\omega-$$
$$-\int_\tau \frac{\partial \sigma_{kr}}{\partial x_r}\delta u_k\,d\tau.$$

Here the index k is summed in the first and second integrands, and the indices k and r are summed in the third integrand.

By virtue of the last relation we have

$$\int_\tau \left\{\left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_r}+\frac{\partial u_r}{\partial x_k}\right)-\frac{\partial A}{\partial \sigma_{kr}^*}\right]\delta\sigma_{kr}-\frac{\partial \sigma_{kr}}{\partial x_r}\delta u_k\right\}d\tau+$$
$$+\int_{\omega_T}(T_{nk}-\overline{T}_{nk})\,\delta u_k\,d\omega+\int_{\omega_u} T_{nk}\delta u_k\,d\omega=0.$$

The last integral is zero because, by condition, $\delta u_k = 0$ on ω_u.

Since the variations of displacements and stresses are arbitrary and independent, on the basis of the fundamental lemma of the calculus of variations we infer from the foregoing condition that the factors of the corresponding variations are zero both in the volume and surface integrals, giving Eqs. (8.31), (8.32) and the boundary conditions (8.33), (8.34).

74. EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS FOR A GEOMETRICALLY NON-LINEAR BODY

In solving some problems of the theory of elasticity, such as stability problems, it is necessary to take into account the components of the finite strain tensor defined by formulas (3.17). Here we shall restrict ourselves to the derivation of the equilibrium equations and boundary conditions for this case.

From the variational equation of equilibrium we derive the equilibrium equations and boundary conditions for the case when the components of the strain tensor are given in the Cartesian co-ordinate system (3.24):

$$\varepsilon_{nk}=\frac{1}{2}\left(\frac{\partial u_n}{\partial x_k}+\frac{\partial u_k}{\partial x_n}+\frac{\partial u_\alpha}{\partial x_k}\frac{\partial u_\alpha}{\partial x_n}\right). \tag{8.35}$$

Suppose that a body is in equilibrium under the action of a surface force T_ν and a volume force ρF. The variational equation is then of the form

$$\int_\tau \sigma_{nk}\delta\varepsilon_{nk}\,d\tau-\int_\tau \rho F_k\delta u_k-\int_\omega T_{nk}\delta u_k\,d\omega=0. \tag{8.36}$$

By varying expression (8.35), we obtain

$$\delta\varepsilon_{kn}=\frac{1}{2}\left(\delta\frac{\partial u_n}{\partial x_k}+\delta\frac{\partial u_k}{\partial x_n}+\frac{\partial u_\alpha}{\partial x_k}\delta\frac{\partial u_\alpha}{\partial x_n}+\frac{\partial u_\alpha}{\partial x_n}\delta\frac{\partial u_\alpha}{\partial x_k}\right).$$

By direct calculation it is easily found that this expression may be put into the form:

$$\delta\varepsilon_{kn}=\frac{1}{2}\left[\left(\delta_{\alpha k}+\frac{\partial u_\alpha}{\partial x_k}\right)\delta\frac{\partial u_\alpha}{\partial x_n}+\left(\delta_{\alpha n}+\frac{\partial u_\alpha}{\partial x_n}\right)\delta\frac{\partial u_\alpha}{\partial x_k}\right]. \tag{8.37}$$

Here $\delta_{\alpha k}$ are the Kronecker symbols.

By (8.37), the variational equation becomes

$$\frac{1}{2}\int_{\tau}\sigma_{nk}\left(\delta_{\alpha k}+\frac{\partial u_{\alpha}}{\partial x_k}\right)\delta\frac{\partial u_{\alpha}}{\partial x_n}\,d\tau+\frac{1}{2}\int_{\tau}\sigma_{nk}\left(\delta_{\alpha n}+\frac{\partial u_{\alpha}}{\partial x_n}\right)\delta\frac{\partial u_{\alpha}}{\partial x_k}\,d\tau-$$

$$-\int_{\tau}\rho F_k\delta u_k-\int_{\omega}T_{\nu k}\delta u_k\,d\omega=0. \quad (8.38)$$

We introduce a non-symmetrical tensor of the form

$$s_{\alpha n}=\sigma_{nk}\left(\delta_{\alpha k}+\frac{\partial u_{\alpha}}{\partial x_k}\right). \quad (8.39)$$

Taking this into account, the variational equation of equilibrium is transformed into

$$\frac{1}{2}\int_{\tau}s_{\alpha n}\delta\frac{\partial u_{\alpha}}{\partial x_n}\,d\tau+\frac{1}{2}\int_{\tau}s_{\alpha k}\delta\frac{\partial u_{\alpha}}{\partial x_k}\,d\tau-\int_{\tau}\rho F_k\delta u_k-\int_{\omega}T_{\nu k}\delta u_k\,d\omega=0.$$

The last equality is represented as

$$\frac{1}{2}\int_{\tau}\frac{\partial}{\partial x_n}(s_{\alpha n}\delta u_{\alpha})\,d\tau-\frac{1}{2}\int_{\tau}\frac{\partial s_{\alpha n}}{\partial x_n}\delta u_{\alpha}\,d\tau+\frac{1}{2}\int_{\tau}\frac{\partial}{\partial x_k}(s_{\alpha k}\delta u_{\alpha})\,d\tau-$$

$$-\frac{1}{2}\int_{\tau}\frac{\partial s_{\alpha k}}{\partial x_k}\delta u_{\alpha}\,d\tau-\int_{\tau}\rho F_k\delta u_k\,d\tau-\int_{\omega}T_{\nu k}\delta u_k\,d\omega=0.$$

By applying the Gauss-Ostrogradsky formula, we find

$$\int_{\omega}(s_{\alpha n}\nu_n-T_{\nu\alpha})\,\delta u_{\alpha}\,d\omega-\int_{\tau}\left(\frac{\partial s_{\alpha n}}{\partial x_n}+\rho F_{\alpha}\right)\delta u_{\alpha}\,d\tau=0.$$

On the basis of the fundamental lemma of the calculus of variations we have

$$\frac{\partial s_{\alpha n}}{\partial x_n}+\rho F_{\alpha}=0, \quad s_{\alpha n}\nu_n-T_{\nu\alpha}=0.$$

With (8.39), we obtain the equilibrium equations

$$\frac{\partial}{\partial x_n}\left[\sigma_{nk}\left(\delta_{\alpha k}+\frac{\partial u_{\alpha}}{\partial x_k}\right)\right]+\rho F_{\alpha}=0 \quad (8.40)$$

and the boundary conditions

$$s_{\alpha n}\nu_n=T_{\nu\alpha}. \quad (8.41)$$

CHAPTER IX

Three-dimensional static problems

For the solution of three-dimensional static problems of the theory of elasticity we have no such efficient analytic techniques as in the plane theory of elasticity. Here we shall consider certain particular solutions of the equilibrium equation in the absence of body forces, for which the displacement increases indefinitely near specific points. These points must lie outside the body or must be contained in special cavities within it. It should be noted that the simplest type of isolated singular point is the point of application of a concentrated force.

75. KELVIN'S AND BOUSSINESQ-PAPKOVICH SOLUTIONS

If a body is acted on by mass forces, the vector equilibrium equation is of the form of (5.7). Suppose that the region occupied by the body extends to infinity in all directions and the mass force $\boldsymbol{F}$ is different from zero in a region τ_1 coinciding either with the whole of the region τ or with a part of it.

We present the general form of the particular solution given by Kelvin (W. Thomson). The displacement vector is expressed in terms of the scalar potential φ and the vector potential $\boldsymbol{\psi}$ by the formula

$$\boldsymbol{u} = \nabla\varphi + \operatorname{rot}\boldsymbol{\psi}. \tag{9.1}$$

Here ∇ is the Hamiltonian operator.

Suppose further that the mass forces may be represented as

$$\boldsymbol{F} = \nabla\Phi + \operatorname{rot}\boldsymbol{\Psi}. \tag{9.2}$$

By using the vector identity

$$\operatorname{rot}\operatorname{rot}\boldsymbol{u} = \nabla\operatorname{div}\boldsymbol{u} - \Delta\boldsymbol{u}$$

n the equilibrium equation (5.7), we come to the equation

$$(\lambda + 2\mu)\nabla\operatorname{div}\boldsymbol{u} - \mu\operatorname{rot}\operatorname{rot}\boldsymbol{u} + \rho\boldsymbol{F} = 0. \tag{9.3}$$

From (9.1) we calculate

$$\nabla\operatorname{div}\boldsymbol{u} = \nabla\Delta\varphi,$$

$$\text{rot rot } \boldsymbol{u} = \text{rot rot } (\nabla\varphi + \text{rot } \boldsymbol{\psi}) = \text{rot } (\text{rot rot } \boldsymbol{\psi}) =$$

$$= \text{rot } (\nabla \operatorname{div} \boldsymbol{\psi} - \Delta\boldsymbol{\psi}) = -\text{rot } \Delta\boldsymbol{\psi}.$$

Substituting these relations and (9.2) in the equilibrium equation (9.3), we find

$$\nabla [(\lambda + 2\mu)\, \Delta\varphi + \rho\Phi] + \text{rot } [\mu\Delta\boldsymbol{\psi} + \rho\boldsymbol{\Psi}] = 0.$$

This equation is satisfied if we assume

$$\Delta\varphi = -\frac{\rho}{\lambda+2\mu}\Phi, \quad \Delta\boldsymbol{\psi} = -\frac{\rho}{\mu}\boldsymbol{\Psi}. \tag{9.4}$$

Thus, a particular solution of Eq. (9.3) can be obtained from particular solutions of Poisson's equations (9.4), which, as is known from potential theory, are of the form

$$\varphi(\boldsymbol{r}) = \frac{\rho}{4\pi(\lambda+2\mu)} \int\limits_{\tau_1} \frac{\Phi(\boldsymbol{r}')}{l}\, d\tau_1, \tag{9.5}$$

$$\boldsymbol{\psi}(\boldsymbol{r}) = \frac{\rho}{4\pi\mu} \int\limits_{\tau_1} \frac{\boldsymbol{\Psi}(\boldsymbol{r}')}{l}\, d\tau_1, \tag{9.6}$$

where $l = [(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2]^{1/2}$ is the distance from a point $\boldsymbol{r}'$ (x_1', x_2', x_3') of the region τ_1 to the point $\boldsymbol{r}$ (x_1, x_2, x_3) for which the functions φ and $\boldsymbol{\psi}$ are calculated; the integrals are extended over the region τ_1 outside whose boundary the mass forces are zero (Fig. 39); the functions $\Phi(\boldsymbol{r})$ and $\boldsymbol{\Psi}(\boldsymbol{r})$ are determined by the formulas

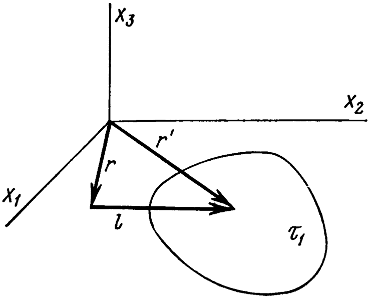

Fig. 39

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi} \int\limits_{\tau_1} \boldsymbol{F} \cdot \nabla l^{-1}\, d\tau_1, \tag{9.7}$$

$$\boldsymbol{\Psi}(\boldsymbol{r}) = -\frac{1}{4\pi} \int\limits_{\tau_1} \boldsymbol{F} \times \nabla l^{-1}\, d\tau_1, \tag{9.8}$$

which follow from the condition that the mass force $\boldsymbol{F}$ may be represented in the form of (9.2). Indeed, with div rot $\boldsymbol{\Psi} = 0$, from (9.2) we find

$$\operatorname{div} \boldsymbol{F} = \operatorname{div} \nabla\Phi, \qquad \text{rot } \boldsymbol{F} = \text{rot rot } \boldsymbol{\Psi},$$

or with div $\boldsymbol{\Psi} = 0$,

$$\Delta\Phi = \frac{\partial F_1}{\partial x_1'} + \frac{\partial F_2}{\partial x_2'} + \frac{\partial F_3}{\partial x_3'},$$

$$\Delta\Psi_1 = -\left(\frac{\partial F_3}{\partial x_2'} - \frac{\partial F_2}{\partial x_3'}\right), \quad \Delta\Psi_2 = -\left(\frac{\partial F_1}{\partial x_3'} - \frac{\partial F_3}{\partial x_1'}\right),$$

$$\Delta\Psi_3 = -\left(\frac{\partial F_2}{\partial x_1'} - \frac{\partial F_1}{\partial x_2'}\right).$$

The particular solutions of these equations may be written as

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= -\frac{1}{4\pi}\int_{\tau_1}\left(\frac{\partial F_1}{\partial x_1'}+\frac{\partial F_2}{\partial x_2'}+\frac{\partial F_3}{\partial x_3'}\right)\frac{1}{l}\,d\tau_1,\\
\Psi_1(\boldsymbol{r}) &= \frac{1}{4\pi}\int_{\tau_1}\left(\frac{\partial F_3}{\partial x_2'}-\frac{\partial F_2}{\partial x_3'}\right)\frac{1}{l}\,d\tau\ ,\\
\Psi_2(\boldsymbol{r}) &= \frac{1}{4\pi}\int_{\tau_1}\left(\frac{\partial F_1}{\partial x_3'}-\frac{\partial F_3}{\partial x_1'}\right)\frac{1}{l}\,d\tau_1,\\
\Psi_3(\boldsymbol{r}) &= \frac{1}{4\pi}\int_{\tau_1}\left(\frac{\partial F_2}{\partial x_1'}-\frac{\partial F_1}{\partial x_2'}\right)\frac{1}{l}\,d\tau_1.
\end{aligned} \tag{9.9}$$

By applying the Gauss-Ostrogradsky formula to the first equality of (9.9), we find

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{\omega}\frac{1}{l}F_k n_k\,d\omega+\frac{1}{4\pi}\int_{\tau_1}F_k\frac{\partial l^{-1}}{\partial x_k'}\,d\tau_1, \tag{9.10}$$

where ω is the surface of the region τ_1.

Assuming that the mass force is continuous in the region τ up to its boundary $\left(\text{then on this boundary } F_k = 0 \text{ and also } \frac{\partial l^{-1}}{\partial x_k'} = -\frac{\partial l^{-1}}{\partial x_k}\right)$, instead of (9.10) we obtain (9.7). Likewise, from the three remaining equations of (9.9) we find

$$\begin{aligned}
\Psi_1(\boldsymbol{r}) &= \frac{1}{4\pi}\int_{\tau_1}\left(F_3\frac{\partial l^{-1}}{\partial x_2}-F_2\frac{\partial l^{-1}}{\partial x_3}\right)d\tau_1,\\
\Psi_2(\boldsymbol{r}) &= \frac{1}{4\pi}\int_{\tau_1}\left(F_1\frac{\partial l^{-1}}{\partial x_3}-F_3\frac{\partial l^{-1}}{\partial x_1}\right)d\tau_1,\\
\Psi_3(\boldsymbol{r}) &= \frac{1}{4\pi}\int_{\tau_1}\left(F_2\frac{\partial l^{-1}}{\partial x_1}-F_1\frac{\partial l^{-1}}{\partial x_2}\right)d\tau_1.
\end{aligned}$$

These three scalar equalities are equivalent to one vector equality (9.8).

To obtain the Boussinesq-Papkovich solution, we represent the general solution of the equilibrium equation (9.3) as

$$\boldsymbol{u} = A\nabla(\varphi+\boldsymbol{r}\cdot\boldsymbol{\psi})+B\boldsymbol{\psi}, \tag{9.11}$$

where $\boldsymbol{r}$ is the radius vector of a point of the body, A and B are unknown constants, φ and $\boldsymbol{\psi}$ are unknown functions of position.

By applying the operator rot rot to both sides of equality (9.11), and taking into account the vector identities rot rot $= \nabla$ div $-\Delta$, rot $\nabla = 0$, we find

$$\text{rot rot}\,\boldsymbol{u} = B\,(\nabla\,\text{div}\,\boldsymbol{\psi} - \Delta\boldsymbol{\psi}).$$

Since

$$\text{div}\,\nabla = \Delta, \quad \Delta\,(\boldsymbol{r}\cdot\boldsymbol{\psi}) = \boldsymbol{r}\cdot\Delta\boldsymbol{\psi} + 2\,\text{div}\,\boldsymbol{\psi}, \tag{9.12}$$

from (9.11) we have

$$\text{div}\,\boldsymbol{u} = A\Delta\varphi + A\boldsymbol{r}\cdot\Delta\boldsymbol{\psi} + (2A + B)\,\text{div}\,\boldsymbol{\psi}. \tag{9.13}$$

Inserting (9.12), (9.13), and $\text{rot rot}\,\boldsymbol{u} = B\,(\nabla\,\text{div}\,\boldsymbol{\psi} - \Delta\boldsymbol{\psi})$ in Eq. (9.3), we have

$$(\lambda + 2\mu)\,A\nabla\,(\Delta\varphi + \boldsymbol{r}\cdot\Delta\boldsymbol{\psi}) + [(\lambda + 2\mu)\,(B + 2A) - \\ - \mu B]\,\nabla\,\text{div}\,\boldsymbol{\psi} + \mu B\Delta\boldsymbol{\psi} + \rho\boldsymbol{F} = 0.$$

This equation is satisfied if we assume

$$\begin{gathered} \Delta\varphi + \boldsymbol{r}\cdot\Delta\boldsymbol{\psi} = 0, \\ (\lambda + 2\mu)\,(B + 2A) - \mu B = 0, \\ B\Delta\boldsymbol{\psi} + \frac{\rho}{\mu}\boldsymbol{F} = 0. \end{gathered} \tag{9.14}$$

From the second equation of (9.14) we find

$$B = -\frac{2\,(\lambda + 2\mu)}{\lambda + \mu}A = -4\,(1 - \nu)\,A.$$

On putting $A = 1$, from the third equation we have

$$\Delta\boldsymbol{\psi} = \frac{\rho}{4\,(1 - \nu)\,\mu}\boldsymbol{F}. \tag{9.15}$$

Substituting (9.15) in the first equation of (9.14), we obtain

$$\Delta\varphi = -\frac{\rho}{4\,(1 - \nu)\,\mu}\boldsymbol{r}\cdot\boldsymbol{F}. \tag{9.16}$$

Thus, the solution of the equilibrium equation (9.3) can be found in the form of (9.11) if the vector function $\boldsymbol{\psi}$ and the scalar function φ satisfy, respectively, Poisson's equations (9.15) and (9.16). The Boussinesq-Papkovich solution involves four scalar functions, namely the scalar function φ and three projections of the vector $\boldsymbol{\psi}$. The representation in which φ is not a harmonic, but a biharmonic function was given by J. Boussinesq, and independently by B. G. Galerkin.

Some problems can be solved without using so many functions. By taking, for example, $\boldsymbol{\psi} = 0$ in solution (9.11), we obtain a simple solution of the form

$$\boldsymbol{u} = A\nabla\varphi. \tag{9.17}$$

By the first equation of (9.14), the function φ is harmonic; moreover, from the third equation of (9.14) it follows that solution (9.17) is suitable for the case when body forces are absent.

From (9.17) we find that

$$\operatorname{div} \boldsymbol{u} = A \operatorname{div} \nabla\varphi = A\Delta\varphi = 0.$$

Thus, for the simple solution of the form of (9.17) the volume strain is identically zero.

By formulas (4.35), for the solution of the form of (9.17) the formulas of the stress tensor become

$$\sigma_{kr} = 2\mu A \frac{\partial^2\varphi}{\partial x_k \, \partial x_r}. \tag{9.18}$$

76. BOUSSINESQ'S ELEMENTARY SOLUTIONS OF THE FIRST AND SECOND KIND

In this section Kelvin's solution will be used to obtain a solution for the case of a concentrated force $\boldsymbol{F}_3$ applied to a solid at the origin of co-ordinates and acting in the x_3 direction.

We take an arbitrarily small neighbourhood of the point of application of the force (the simplest singular point) bounded by the planes $x_k = \pm\frac{1}{2}\varepsilon$, and assume that $F = \frac{F_3}{\rho\varepsilon^3}$; from (9.7) and (9.8) we then find

$$\Phi(\boldsymbol{r}) = \frac{F_3}{4\pi\rho}\frac{x_3}{r^3}, \quad \boldsymbol{\Psi}(\boldsymbol{r}) = \frac{F_3}{4\pi\rho}\left(-\boldsymbol{i}_1\frac{x_2}{r^3} + \boldsymbol{i}_2\frac{x_1}{r^3}\right). \tag{9.19}$$

Substituting (9.19) in (9.4), we obtain

$$\Delta\varphi + \frac{F_3}{4\pi(\lambda+2\mu)}\frac{x_3}{r^3} = 0, \quad \Delta\boldsymbol{\psi} + \frac{F_3}{4\pi\mu}\left(-\boldsymbol{i}_1\frac{x_2}{r^3} + \boldsymbol{i}_2\frac{x_1}{r^3}\right) = 0.$$

Since $\Delta\left(\frac{x_k}{r}\right) = -\frac{2x_k}{r^3}$, the last equations may be written as

$$\Delta\left(\varphi - \frac{F_3 x_3}{8\pi(\lambda+2\mu)\,r}\right) = 0,$$

$$\Delta\left[\boldsymbol{\psi} + \frac{F_3}{8\pi\mu}\left(\boldsymbol{i}_1\frac{x_2}{r} - \boldsymbol{i}_2\frac{x_1}{r}\right)\right] = 0.$$

These equations are satisfied if

$$\varphi(\boldsymbol{r}) = \frac{F_3 x_3}{8\pi(\lambda+2\mu)\,r}, \quad \boldsymbol{\psi}(\boldsymbol{r}) = \frac{F_3}{8\pi\mu}\left(-\boldsymbol{i}_1\frac{x_2}{r} + \boldsymbol{i}_2\frac{x_1}{r}\right),$$

from which

$$\nabla\varphi = \frac{F_3}{8\pi(\lambda+2\mu)}\left[-\boldsymbol{i}_1\frac{x_1x_3}{r^3} - \boldsymbol{i}_2\frac{x_2x_3}{r^3} + \boldsymbol{i}_3\left(\frac{1}{r} - \frac{x_3^2}{r^3}\right)\right],$$

$$\operatorname{rot}\boldsymbol{\psi} = \frac{F_3}{8\pi\mu}\left[\boldsymbol{i}_1\frac{x_1x_3}{r^3} + \boldsymbol{i}_2\frac{x_2x_3}{r^3} + \boldsymbol{i}_3\left(\frac{1}{r} + \frac{x_3^2}{r^3}\right)\right].$$

Substituting these expressions in formula (9.1), we find

$$\boldsymbol{u}=\frac{(\lambda+\mu)F_3}{8\pi\mu(\lambda+2\mu)}\left[\boldsymbol{i}_1\frac{x_1x_3}{r^3}+\boldsymbol{i}_2\frac{x_2x_3}{r^3}+\boldsymbol{i}_3\left(\frac{x_3^2}{r^3}+\frac{\lambda+3\mu}{\lambda+\mu}\,\frac{1}{r}\right)\right]. \tag{9.20}$$

Formula (9.20), obtained from Kelvin's solution as a special example, was first derived by J. Boussinesq and designated as an elementary solution of the first kind.

From (9.20) and the formulas of Hooke's law we have the following relations for six components of the stress tensor:

$$\begin{gathered}
\sigma_{11}=B\frac{x_3}{r^3}\left[3\left(\frac{x_1}{r}\right)^2-\frac{\mu}{\lambda+\mu}\right],\quad \sigma_{12}=3B\frac{x_1x_2x_3}{r^5},\\
\sigma_{22}=B\,\frac{x_3}{r^3}\left[3\left(\frac{x_2}{r}\right)^2-\frac{\mu}{\lambda+\mu}\right],\quad \sigma_{23}=B\frac{x_2}{r^3}\left[3\left(\frac{x_3}{r}\right)^2+\frac{\mu}{\lambda+\mu}\right],\\
\sigma_{33}=B\frac{x_3}{r^3}\left[3\left(\frac{x_3}{r}\right)^2+\frac{\mu}{\lambda+\mu}\right],\quad \sigma_{31}=B\frac{x_1}{r^3}\left[3\left(\frac{x_3}{r}\right)^2+\frac{\mu}{\lambda+\mu}\right],
\end{gathered} \tag{9.21}$$

where $B=-\frac{(\lambda+\mu)F_3}{4\pi(\lambda+2\mu)}$.

By an elementary solution of the second kind is meant solution (9.17) in which the harmonic function φ is equal to

$$\varphi=A\ln(r+x_3).$$

Then

$$u_1=A\frac{x_1}{r(x_3+r)},\quad u_2=A\frac{x_2}{r(x_3+r)},\quad u_3=A\frac{1}{r}. \tag{9.22}$$

On the basis of formulas (9.18) we find

$$\begin{gathered}
\sigma_{11}=2\mu A\left(\frac{x_2^2+x_3^2}{r^3(x_3+r)}-\frac{x_1^2}{r^2(x_3+r)^2}\right),\\
\sigma_{22}=2\mu A\left(\frac{x_1^2+x_3^2}{r^3(x_3+r)}-\frac{x_2^2}{r^2(x_3+r)^2}\right),\\
\sigma_{33}=-2\mu A\frac{x_3}{r^3},\quad \sigma_{12}=-2\mu A\frac{x_1x_2(x_3+2r)}{r^3(x_3+r)^2},\\
\sigma_{23}=-2\mu A\frac{x_2}{r^3},\quad \sigma_{31}=-2\mu A\frac{x_1}{r^3}.
\end{gathered} \tag{9.23}$$

The components of the stress vector acting on a plane passing through a given point perpendicular to the radius vector $\boldsymbol{r}$ are, by formulas (9.22), with (9.23), and also with the use of the fact that $\cos(r,\ x_k)=\frac{x_k}{r}$,

$$\begin{gathered}
T_{r1}=-2\mu A\frac{x_1}{r^2(x_3+r)},\\
T_{r2}=-2\mu A\frac{x_2}{r^2(x_3+r)},\quad T_{r3}=-2\mu A\frac{1}{r^2}.
\end{gathered} \tag{9.24}$$

From the singular point (the origin of co-ordinates) describe a sphere of radius $\varepsilon > 0$, and consider its part situated in the region $x_3 > 0$. On the basis of formulas (9.24) it may be stated, without calculations, that the resultant moment of all forces acting on the surface of the hemisphere and the projections of the resultant vector of these forces on the x_1 and x_2 axes are zero, and the projection of the resultant vector on the x_3 axis is

$$R_3 = \int_{\omega} T_{r3}\, d\omega = -2\mu A \frac{1}{\varepsilon^2} \int_{\omega} d\omega = -4\pi\mu A.$$

Consequently, the resultant of all forces acting on the surface of the hemisphere is directed along the x_3 axis and is of magnitude

$$R_3 = -4\pi\mu A. \tag{9.25}$$

Thus, the origin represents a simple type of isolated singular point at which is applied a concentrated force directed along the ox_3 axis and of magnitude $F_3 = 2R_3 = -8\pi\mu A$.

77. PRESSURE ON THE SURFACE OF A SEMI-INFINITE BODY

In order to find the field of the stress tensor in a body occupying the half-space $x_3 > 0$ subjected to a concentrated force $\boldsymbol{T}$ applied normally to the plane boundary x_1x_2 of this body, we make use of the results of the preceding sections. Transfer the origin to the point of application of this force.

Take a solution of Eq. (9.3) as the sum of solutions (9.20) and (9.22). Then

$$\begin{aligned} u_1 &= \frac{(\lambda+\mu) F_3}{8\pi(\lambda+2\mu)\mu} \frac{x_1 x_3}{r^3} + A \frac{x_1}{r(x_3+r)}, \\ u_2 &= \frac{(\lambda+\mu) F_3}{8\pi(\lambda+2\mu)\mu} \frac{x_2 x_3}{r^3} + A \frac{x_2}{r(x_3+r)}, \\ u_3 &= \frac{(\lambda+\mu) F_3}{8\pi(\lambda+2\mu)\mu} \left[\frac{x_3^2}{r^3} + \frac{\lambda+3\mu}{\lambda+\mu} \frac{1}{r} \right] + A \frac{1}{r}. \end{aligned} \tag{9.26}$$

These relations represent a solution of Eq. (9.3) everywhere, except at the point of application of the force $\boldsymbol{T}$.

We shall try to determine F_3 and A so as to fulfil the conditions of zero external forces on the boundary $x_3 = 0$, i.e., $\boldsymbol{T}_{33} = \boldsymbol{T}_{31} = \boldsymbol{T}_{32} = 0$; hence,

$$\sigma_{33} = \sigma_{31} = \sigma_{32} = 0; \tag{9.27}$$

by the laws of statics, at the point of application of the force

$$T = \frac{F_3}{2} + 4\pi\mu A. \tag{9.28}$$

By using formulas (9.21) and (9.23), from (9.27) we obtain

$$\sigma_{31}=-\frac{2x_1}{r^3}\mu\left[\frac{F_3}{8\pi(\lambda+2\mu)}+A\right]=0,\quad \sigma_{33}=0,$$
$$\sigma_{32}=-\frac{2x_2}{r^3}\mu\left[\frac{F_3}{8\pi(\lambda+2\mu)}+A\right]=0,$$

from which

$$\frac{1}{8\pi(\lambda+2\mu)}F_3+A=0. \tag{9.29}$$

Thus, for the determination of F_3 and A we have obtained two equations (9.28) and (9.29), from which we find

$$F_3=\frac{2(\lambda+2\mu)T}{\lambda+\mu},\quad A=-\frac{T}{4\pi(\lambda+\mu)}.$$

Substituting the values obtained for F_3 and A in formulas (9.26), we find Boussinesq's formulas

$$\begin{aligned}u_1&=\frac{T}{4\pi\mu}\frac{x_1x_3}{r^3}-\frac{T}{4\pi(\lambda+\mu)}\frac{x_1}{r(x_3+r)},\\ u_2&=\frac{T}{4\pi\mu}\frac{x_2x_3}{r^3}-\frac{T}{4\pi(\lambda+\mu)}\frac{x_2}{r(x_3+r)},\\ u_3&=\frac{T}{4\pi\mu}\frac{x_3^2}{r^3}+\frac{T(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\frac{1}{r}.\end{aligned} \tag{9.30}$$

Solutions (9.30) give the values for displacements at all points of an elastic body sufficiently far from the point of application of the force $\boldsymbol{T}$.

At points of the boundary ox_1x_2, where $x_3=0$, the displacements are determined by the formulas

$$\begin{aligned}u_1&=-\frac{T}{4\pi(\lambda+\mu)}\frac{x_1}{r^2},\\ u_2&=-\frac{T}{4\pi(\lambda+\mu)}\frac{x_2}{r^2},\\ u_3&=\frac{T(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\frac{1}{r},\end{aligned} \tag{9.31}$$

where $r=\sqrt{x_1^2+x_2^2}$.

Inserting the values of F_3 and A in formulas (9.21) and (9.23), the field of the stress tensor in the half-space under consideration is determined by

$$\sigma_{11}=-\frac{Tx_3}{2\pi r^3}\left[3\left(\frac{x_1}{r}\right)^2-\frac{\mu}{\lambda+\mu}\right]-\frac{\mu T}{2\pi(\lambda+\mu)}\left[\frac{x_2^2+x_3^2}{r^3(r+x_3)}-\frac{x_1^2}{r^2(r+x_3)^2}\right],$$
$$\sigma_{22}=-\frac{Tx_3}{2\pi r^3}\left[3\left(\frac{x_2}{r}\right)^2-\frac{\mu}{\lambda+\mu}\right]-\frac{\mu T}{2\pi(\lambda+\mu)}\left[\frac{x_1^2+x_3^2}{r^3(r+x_3)}-\frac{x_2^2}{r^2(r+x_3)^2}\right],$$

$$\sigma_{33} = -\frac{Tx_3}{2\pi r^3}\left[3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right] + \frac{\mu T}{2\pi(\lambda+\mu)}\frac{x_3}{r^3},$$

$$\sigma_{12} = -\frac{3Tx_1x_2x_3}{2\pi r^5} + \frac{\mu T}{2\pi(\lambda+\mu)}\frac{x_1x_2(x_3+2r)}{r^3(r+x_3)^2},$$

$$\sigma_{23} = -\frac{Tx_2}{2\pi r^3}\left[3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right] + \frac{\mu T}{2\pi(\lambda+\mu)}\frac{x_2}{r^3},$$

$$\sigma_{31} = -\frac{Tx_1}{2\pi r^3}\left[3\left(\frac{x_3}{r}\right)^2 + \frac{\mu}{\lambda+\mu}\right] + \frac{\mu T}{2\pi(\lambda+\mu)}\frac{x_1}{r^3}.$$

Let $q(\xi, \eta)$ be the intensity of a force distributed over some area ω of the boundary plane ox_1x_2 of the hemisphere. The element of area $d\xi\, d\eta$ is acted on by the force

$$dT = q(\xi, \eta)\, d\xi\, d\eta;$$

on the basis of solution (9.30) the displacements are

$$\begin{aligned} u_1 &= \frac{1}{4\pi\mu}\int_\omega \left(\frac{x_1x_3}{r^3} - \frac{\mu}{\lambda+\mu}\frac{x_1}{r(x_3+r)}\right) q(\xi, \eta)\, d\xi\, d\eta, \\ u_2 &= \frac{1}{4\pi\mu}\int_\omega \left(\frac{x_2x_3}{r^3} - \frac{\mu}{\lambda+\mu}\frac{x_2}{r(x_3+r)}\right) q(\xi, \eta)\, d\xi\, d\eta, \\ u_3 &= \frac{1}{4\pi\mu}\int_\omega \left(\frac{x_3^2}{r^3} + \frac{\lambda+2\mu}{\lambda+\mu}\frac{1}{r}\right) q(\xi, \eta)\, d\xi\, d\eta, \end{aligned} \tag{9.32}$$

where

$$r = \sqrt{(x_1-\xi)^2 + (x_2-\eta)^2 + x_3^2}.$$

Here ξ, η are the co-ordinates of the point of application of the force dT; x_1, x_2, x_3 are the co-ordinates of the point at which the displacements u_1, u_2, u_3 are sought.

The displacement along the x_3 axis of any point of the boundary ox_1x_2 is, according to (9.32),

$$u_3 = \theta \int_\omega \frac{q(\xi, \eta)}{\sqrt{(x_1-\xi)^2 + (x_2-\eta)^2}}\, d\xi\, d\eta, \tag{9.33}$$

where

$$\theta = \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}.$$

78. HERTZ'S PROBLEM OF THE PRESSURE BETWEEN TWO BODIES IN CONTACT

Suppose that two homogeneous isotropic bodies, *1* and *2*, with different elastic constants are in contact at a point *o*, which is taken to be the origin of a rectangular Cartesian co-ordinate system $x_1x_2x_3$. Let the axes ox_1, ox_2 be placed in a plane tangential to both bodies at the point o, and let the axes $x_3^{(1)}$, $x_3^{(2)}$ be taken coincident, respectively, with the inward normals to the surfaces of these bodies (Fig. 40). Referred to these co-ordinate systems, the equations of the surfaces of the contacting bodies before deformation are

$$x_3^{(1)}=f_1(x_1, x_2), \quad x_3^{(2)}=f_2(x_1, x_2). \tag{9.34}$$

Equations (9.34) of the surfaces of the bodies near their point of contact o (the point o is supposed to be regular) may be represented with sufficient accuracy as

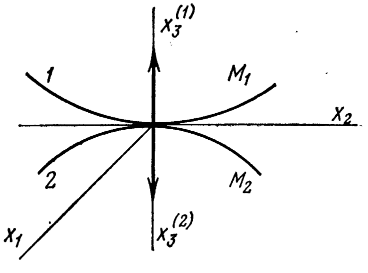

Fig. 40

$$x_3^{(1)}=\frac{1}{2}\left.\frac{\partial^2 x_3^{(1)}}{\partial x_1^2}\right|_{0,0} x_1^2+\frac{1}{2}\left.\frac{\partial^2 x_3^{(1)}}{\partial x_2^2}\right|_{0,0} x_2^2+\left.\frac{\partial^2 x_3^{(1)}}{\partial x_1 \partial x_2}\right|_{0,0} x_1 x_2,$$

$$x_3^{(2)}=\frac{1}{2}\left.\frac{\partial^2 x_3^{(2)}}{\partial x_1^2}\right|_{0,0} x_1^2+\frac{1}{2}\left.\frac{\partial^2 x_3^{(2)}}{\partial x_2^2}\right|_{0,0} x_2^2+\left.\frac{\partial^2 x_3^{(2)}}{\partial x_1 \partial x_2}\right|_{0,0} x_1 x_2.$$

The distance between two points, M_1 and M_2, of the contacting surfaces lying on the same normal to the tangential plane ox_1x_2 is determined, according to the last relations, by the formula

$$x_3^{(1)}+x_3^{(2)}=(A_1+A_2)x_1^2+(B_1+B_2)x_2^2+(H_1+H_2)x_1x_2. \tag{9.35}$$

Here the following notation has been introduced:

$$A_1=\frac{1}{2}\left.\frac{\partial^2 x_3^{(1)}}{\partial x_1^2}\right|_{0,0}, \quad A_2=\frac{1}{2}\left.\frac{\partial^2 x_3^{(2)}}{\partial x_1^2}\right|_{0,0},$$

$$B_1=\frac{1}{2}\left.\frac{\partial^2 x_3^{(1)}}{\partial x_2^2}\right|_{0,0},$$

$$B_2=\frac{1}{2}\left.\frac{\partial^2 x_3^{(2)}}{\partial x_2^2}\right|_{0,0}, \quad H_1=\left.\frac{\partial^2 x_3^{(1)}}{\partial x_1 \partial x_2}\right|_{0,0}, \quad H_2=\left.\frac{\partial^2 x_3^{(2)}}{\partial x_1 \partial x_2}\right|_{0,0}.$$

The quadratic form $(A_1 + A_2)\, x_1^{(2)} + (B_1 + B_2)\, x_2^2 + (H_1 + H_2)\, x_1 x_2$ defining the distance between the indicated points M_1 and M_2 must be positive for any choice of the x_1 and x_2 axes, and we can choose the axes so as to make the coefficient $H_1 + H_2$ zero. Introducing the notation $A = A_1 + A_2$, $B = B_1 + B_2$, we have

$$x_3^{(1)} + x_3^{(2)} = Ax_1^2 + Bx_2^2. \tag{9.36}$$

Consequently, the coefficients A and B are positive.

Let the principal radii of curvature at the point of contact for the first body be denoted by $R_1^{(1)}$ and $R_2^{(1)}$, and for the second body by $R_1^{(2)}$ and $R_2^{(2)}$. If they are considered to be positive, then

$$2A = \frac{1}{R_1^{(1)}} + \frac{1}{R_2^{(1)}}, \quad 2B = \frac{1}{R_1^{(2)}} + \frac{1}{R_2^{(2)}}.$$

From (9.36) we draw a conclusion that the curves of equal distance between two points of the contacting surfaces lying on the same normal to the tangential plane ox_1x_2 are concentric ellipses.

Suppose that the two bodies are pressed to each other by a force T directed along the normal to the tangential plane ox_1x_2 at the point o; near this point the bodies make contact over a small surface. This surface is called the pressure surface, and its contour is called the pressure contour. The projection of the pressure surface on the tangential plane ox_1x_2 is termed the region of contact. It may be assumed, with sufficient accuracy, that in compression the bodies come into contact at points lying before deformation on the same normal to the ox_1x_2 plane. It is seen from (9.36) that the pressure surface has an elliptical shape.

As a result of the compression of two bodies any two points, lying on the $x_3^{(1)}$ and $x_3^{(2)}$ axes sufficiently far from the point o for the deformations at them to be neglected, come closer together by an amount α equal to the sum of the displacements $u_{3o}^{(1)}$ and $u_{3o}^{(2)}$ of the point o.

Let $u_3^{(1)}$ and $u_3^{(2)}$ denote, respectively, the displacements of points of the two contacting surfaces lying on the same normal to the ox_1x_2 plane in the directions of the $ox_3^{(1)}$ and $ox_3^{(2)}$ axes. The distance between two such points decreases by an amount equal to $\alpha - (u_3^{(1)} + u_3^{(2)})$. Thus, for all points of the pressure surface the following relation holds:

$$x_3^{(1)} + u_3^{(1)} + x_3^{(2)} + u_3^{(2)} = \alpha, \tag{9.37}$$

and for the points outside the pressure surface we must have

$$x_3^{(1)} + u_3^{(1)} + x_3^{(2)} + u_3^{(2)} > \alpha.$$

By using relation (9.36) in formula (9.37), we find

$$u_3^{(1)} + u_3^{(2)} = \alpha - Ax_1^2 - Bx_2^2. \tag{9.38}$$

To determine the elastic displacements and stresses in the region of contact between the two bodies, we assume that the pressure surface is very small and that the bodies may be replaced by half-spaces. These half-spaces are acted on by a normal pressure $q(\xi, \eta)$ over the region of contact ω; the frictional forces on the pressure surface are neglected, i.e., we assume that there are no shearing stresses in the region of contact.

By using formula (9.33) in (9.38), we obtain

$$\int_{\omega} \frac{q(\xi, \eta)}{r}\, d\xi\, d\eta = (\alpha - Ax_1^2 - Bx_2^2)(\theta_1 + \theta_2)^{-1}. \tag{9.39}$$

Here

$$\theta_1 = \frac{\lambda_1 + 2\mu_1}{4\pi\mu_1(\lambda_1 + \mu_1)}, \quad \theta_2 = \frac{\lambda_2 + 2\mu_2}{4\pi\mu_2(\lambda_2 + \mu_2)};$$

λ_1, μ_1 and λ_2, μ_2 are Lamé's elastic constants of the first and second bodies, respectively; A and B are known positive quantities determined from the shapes of the contacting surfaces.

Thus, the solution of the Hertz contact problem is reduced to the determination of the pressure $q(\xi, \eta)$, the approach of the bodies α, and the size and shape of the region of contact ω. In Eq. (9.39) the value of the convergent improper integral represents the potential for a simple layer distributed with density $q(\xi, \eta)$ over the region of contact. This potential at points of the region of contact represents, according to (9.39), a quadratic function of position. On the other hand, it is known that the potential at interior points of the homogeneous ellipsoid

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

is a quadratic function of the co-ordinates of the point and is of the form

$$\varphi = \pi abc\rho \int_0^{\infty} \frac{\left(1 - \frac{x_1^2}{a^2+\lambda} - \frac{x_2^2}{b^2+\lambda} - \frac{x_3^2}{c^2+\lambda}\right) d\lambda}{[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{1/2}}.$$

On comparing these facts, H. Hertz concludes that the right-hand side of formula (9.39) may be taken as the potential for a homogeneous ellipsoid whose thickness in the ox_3 direction tends to zero ($c \to 0$), and the density ρ increases in proportion, so that the mass of the ellipsoid remains unchanged. The region of contact ω is then an ellipse into which the ellipsoid degenerates as $c \to 0$, and the following relation holds:

$$\frac{1}{\theta_1 + \theta_2}(\alpha - Ax_1^2 - Bx_2^2) = \pi ab \lim (c\rho) \int_0^{\infty} \frac{1 - \frac{x_1^2}{a^2+\lambda} - \frac{x_2^2}{b^2+\lambda}}{[(a^2+\lambda)(b^2+\lambda)\lambda]^{1/2}}\, d\lambda. \tag{9.40}$$

The density of a simple layer $q(\xi, \eta)$ is equal to the mass enclosed in a prism of unit base and of height

$$2c\sqrt{1-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}},$$

i.e., the contact pressure is obtained as

$$q(\xi, \eta)=2\lim(c\rho)\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}}. \tag{9.41}$$

Based on the laws of statics, the force T maintaining the bodies in contact can obviously be obtained as the resultant of all forces $q(\xi, \eta)$ over the region of contact ω. Consequently, it is equal in magnitude to the mass of the whole ellipsoid, i.e.,

$$T=\frac{4}{3}\pi ab\lim(c\rho). \tag{9.42}$$

Eliminating, now, $\lim(c\rho)$ from formulas (9.41) and (9.42), we obtain, finally,

$$q(\xi, \eta)=\frac{3T}{2\pi ab}\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}}. \tag{9.43}$$

From equality (9.40) we find

$$A=\frac{3}{4}T(\theta_1+\theta_2)\int_0^\infty\frac{d\lambda}{(a^2+\lambda)^{3/2}[(b^2+\lambda)\lambda]^{1/2}}, \tag{9.44}$$

$$B=\frac{3}{4}T(\theta_1+\theta_2)\int_0^\infty\frac{d\lambda}{(b^2+\lambda)^{3/2}[(a^2+\lambda)\lambda]^{1/2}}, \tag{9.45}$$

$$\alpha=\frac{3}{4}T(\theta_1+\theta_2)\int_0^\infty\frac{d\lambda}{[(a^2+\lambda)(b^2+\lambda)\lambda]^{1/2}}. \tag{9.46}$$

After determining the semiaxes a and b from the first two equations, we find α from the third equation. In the general case the determination of a, b, and α involves the calculation of the elliptic integrals of the first and second kind.

If the two bodies in contact are spheres, the calculations are simplified. In this case $A=B=\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$, where R_1 and R_2 are the radii of the spheres. Taking into account that $A=B$, and using formulas (9.44) and (9.45), we have $a=b$ (the pressure surface is a circle); consequently, formula (9.44) becomes

$$A=\frac{3T(\theta_1+\theta_2)}{4}\int_0^\infty\frac{d\lambda}{(a^2+\lambda)^2\lambda^{1/2}}.$$

Denoting $\lambda = a^2 \tan^2 \varphi$, we find

$$A = \frac{3T(\theta_1+\theta_2)}{4a^3} \int_0^{\pi/2} 2\cos^2\varphi \, d\varphi$$

or

$$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = \frac{3T(\theta_1+\theta_2)\pi}{8a^3}.$$

Hence,

$$a = \sqrt[3]{\frac{3T(\theta_1+\theta_2)\pi}{4\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}}, \tag{9.47}$$

i.e., the radius of the pressure circle is proportional to the cubic root of the force T.

From formulas (9.43) and (9.46) we have, respectively,

$$q(\xi, \eta) = \frac{3T}{2\pi a^3}\sqrt{a^2-(\xi^2+\eta^2)}, \tag{9.48}$$

$$\alpha = \frac{3}{2} T(\theta_1+\theta_2)\frac{1}{a}\int_0^{\pi/2} d\varphi = \frac{3}{4a} T(\theta_1+\theta_2)\pi$$

or

$$\alpha = \sqrt[3]{\frac{9\pi^2}{16} T^2(\theta_1+\theta_2)^2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}. \tag{9.49}$$

In the case of two identical spheres the following equalities hold:

$$R_1 = R_2 = R, \quad \theta_1 = \theta_2 = \theta.$$

On the basis of formulas (9.47) and (9.49) we then have

$$a = \sqrt[3]{\frac{3}{4}\pi\theta T R}, \quad \alpha = \sqrt[3]{\frac{9\pi^2\theta^2 T^2}{2R}}.$$

If the second body is a half-space ($R_2 = \infty$), then

$$a = \sqrt[3]{\frac{3}{4}\pi(\theta_1+\theta_2)TR_1}, \quad \alpha = \sqrt[3]{\frac{9\pi^2}{16R_1}T^2(\theta_1+\theta_2)^2}.$$

When an absolutely rigid plane ($\theta_2 = 0$) is indented by an elastic sphere with a force T, we have

$$a = \sqrt[3]{\frac{3}{4}\pi\theta_1 T R_1}, \quad \alpha = \sqrt[3]{\frac{9\pi^2}{16R_1}\theta_1^2 T^2}.$$

The pressure $q(\xi, \eta)$ is determined in all these cases by formula (9.48).

79. SYMMETRICAL DEFORMATION OF A BODY OF REVOLUTION

Let a body, representing a body of revolution about the x_3 axis, be deformed under the action of surface forces (body forces are absent) symmetrically with respect to this axis of revolution. The displacement in a direction perpendicular to a plane passing through the x_3 axis is zero, and the other two projections, u_r and u_3, are independent of the polar angle φ. For the solution of this problem it is convenient to use cylindrical co-ordinates r, φ, x_3. The components of the symmetrical strain tensor in the cylindrical co-ordinate system are, by formulas (3.29),

$$\begin{gathered} e_{rr}=\frac{\partial u_r}{\partial r}, \quad e_{\varphi\varphi}=\frac{u_r}{r}, \quad e_{33}=\frac{\partial u_3}{\partial x_3}, \\ e_{r3}=\frac{1}{2}\left(\frac{\partial u_r}{\partial x_3}+\frac{\partial u_3}{\partial r}\right), \quad e_{r\varphi}=e_{\varphi 3}=0. \end{gathered} \tag{9.50}$$

Substituting (9.50) in the formulas of Hooke's law, and expressing Lamé's coefficients λ and μ in terms of E and ν, we have

$$\begin{aligned} \sigma_{rr}&=\frac{\nu E}{(1+\nu)(1-2\nu)}\left[\frac{1-\nu}{\nu}\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_3}{\partial x_3}\right], \\ \sigma_{\varphi\varphi}&=\frac{\nu E}{(1+\nu)(1-2\nu)}\left[\frac{\partial u_r}{\partial r}+\frac{1-\nu}{\nu}\frac{u_r}{r}+\frac{\partial u_3}{\partial x_3}\right], \\ \sigma_{33}&=\frac{\nu E}{(1+\nu)(1-2\nu)}\left[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1-\nu}{\nu}\frac{\partial u_3}{\partial x_3}\right], \\ \sigma_{r3}&=\frac{E}{2(1+\nu)}\left(\frac{\partial u_r}{\partial x_3}+\frac{\partial u_3}{\partial r}\right). \end{aligned} \tag{9.51}$$

If we assume

$$\begin{gathered} u_r=-\frac{1+\nu}{E}\frac{\partial^2\Phi}{\partial r\,\partial x_3}, \\ u_3=\frac{1+\nu}{E}\left[(1-2\nu)\Delta\Phi+\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right], \end{gathered} \tag{9.52}$$

formulas (9.51) become

$$\begin{aligned} \sigma_{rr}&=\frac{\partial}{\partial x_3}\left(\nu\Delta\Phi-\frac{\partial^2\Phi}{\partial r^2}\right), \\ \sigma_{\varphi\varphi}&=\frac{\partial}{\partial x_3}\left(\nu\Delta\Phi-\frac{1}{r}\frac{\partial\Phi}{\partial r}\right), \\ \sigma_{33}&=\frac{\partial}{\partial x_3}\left[(2-\nu)\Delta\Phi-\frac{\partial^2\Phi}{\partial x_3^2}\right], \\ \sigma_{r3}&=\frac{\partial}{\partial r}\left[(1-\nu)\Delta\Phi-\frac{\partial^2\Phi}{\partial x_3^2}\right], \end{aligned} \tag{9.53}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x_3^2},$$

$\Phi(r, x_3)$ is the stress function.

Functions (9.53) identically satisfy the first two differential equations of equilibrium of (2.30), and the third equation takes the form

$$\Delta\Delta\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x_3^2}\right)^2 \Phi = 0. \tag{9.54}$$

Under this condition functions (9.53) identically satisfy the compatibility equations (5.37). Thus, the problem of the symmetrical deformation of a body of revolution is reduced to that of finding a solution of the biharmonic equation satisfying the appropriate boundary conditions.

We present the solution of the problem of the symmetrical deformation of a solid circular cylinder produced by forces applied on its lateral surface and symmetrically distributed with respect to its axis. To solve this problem, we determine the stress function Φ from Eq. (9.54). Obviously, a solution of the equation

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial x_3^2} = 0 \tag{9.55}$$

is also a solution of Eq. (9.54). This solution may be taken in the form

$$\Phi^* = \Phi_1(r)\sin kx_3. \tag{9.56}$$

From Eq. (9.55) we then obtain an ordinary differential equation for the function $\Phi_1(r)$

$$\frac{d^2\Phi_1}{dr^2} + \frac{1}{r}\frac{d\Phi_1}{dr} - k^2\Phi_1 = 0. \tag{9.57}$$

Noting that one of the fundamental solutions of Eq. (9.57) becomes infinite when $r = 0$, our interest will be concentrated on the bounded solution, which is of the form

$$\Phi_1 = c_1\left(1 + \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2\cdot 4^2} + \frac{k^6r^6}{2^2\cdot 4^2\cdot 6^2} + \dots\right). \tag{9.58}$$

The series within the parentheses in expression (9.58) is called the Bessel function of zero order with imaginary argument (ikr) and represented by the symbol $I_0(ikr)$; instead of (9.56) we then have

$$\Phi^* = c_1 I_0(ikr)\sin kx_3. \tag{9.59}$$

The derivative of the Bessel function with respect to the imaginary argument (ikr) with a negative sign is called the Bessel function of the first order and represented by the symbol $I_1(ikr)$. By direct checking it can easily be established that the following relation holds:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2\right)\Phi_2(r)=2k^2I_0(ikr)$$

if

$$\Phi_2(r)=r\frac{d}{dr}I_0(ikr)=-ikrI_1(ikr).$$

Noting that the function $I_0(ikr)$ is a solution of Eq. (9.57), we come to the conclusion that the function $\Phi_2(r)$ is a solution of the equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2\right)\left(\frac{d^2\Phi_2}{dr^2}+\frac{1}{r}\frac{d\Phi_2}{dr}-k^2\Phi_2\right)=0.$$

Consequently, the solution of Eq. (9.54) may be represented as

$$\Phi^{**}=c_2\,ikrI_1(ikr)\sin kx_3. \tag{9.60}$$

Thus, on the basis of (9.59) and (9.60) the stress function may be expressed as

$$\Phi=[c_1I_0(ikr)+c_2ikrI_1(ikr)]\sin kx_3. \tag{9.61}$$

Substituting this stress function Φ in formulas (9.52), we find the components of the stress tensor; for example, for σ_{rr} and σ_{r3} we have

$$\begin{aligned}\sigma_{rr}&=[c_1\psi_1(r)+c_2\psi_2(r)]\cos kx_3,\\ \sigma_{r3}&=[c_1\psi_3(r)+c_2\psi_4(r)]\sin kx_3,\end{aligned} \tag{9.62}$$

where $\psi_1(r)$, $\psi_2(r)$, $\psi_3(r)$, $\psi_4(r)$ are completely determined functions expressed in terms of $I_0(ikr)$ and $I_1(ikr)$, which are not presented here.

By (9.62), the boundary conditions on the lateral surface of the cylinder are

$$\begin{aligned}T_{rr}&=[c_1\psi_1(R)+c_2\psi_2(R)]\cos kx_3,\\ T_{r3}&=[c_1\psi_3(R)+c_2\psi_4(R)]\sin kx_3.\end{aligned} \tag{9.63}$$

By a suitable choice of the constants k, c_1, c_2 it is possible to study different kinds of loads symmetrical with respect to the axis of the cylinder and acting on its lateral surface. For example, in the case when the lateral surface of the cylinder is acted on by normal pressures $p_n\cos\frac{n\pi x_3}{L}$ and tangential forces $q_n\sin\frac{n\pi x_3}{L}$ and when $k=\frac{n\pi}{L}$ (L is the length of the cylinder), from formulas (9.63) we find

$$\begin{aligned}c_1\psi_1(R)+c_2\psi_2(R)&=-p_n,\\ c_1\psi_3(R)+c_2\psi_4(R)&=q_n.\end{aligned} \tag{9.64}$$

From this we obtain the values of the constants c_1 and c_2. If the solution of Eq. (9.54) is taken in the form

$$\Phi = [c_3 I_0(ikr) + c_4 ikr I_1(ikr)] \cos kx_3, \tag{9.65}$$

by a suitable choice of the constants c_3, c_4, k we obtain the solution of the problem when the lateral surface of the cylinder is acted on by normal pressures $p_n \sin \frac{n\pi x_3}{L}$ and tangential forces $q_n \cos \frac{n\pi x_3}{L}$.

Thus, on combining solutions (9.61) and (9.65), and using the principle of superposition of the actions of forces, we can obtain any distribution, symmetrical with respect to the axis of the cylinder, of normal and tangential forces on its lateral surface. At the ends of the cylinder there may be some forces symmetrically distributed with respect to the axis of the cylinder. By superimposing an axial tensile or compressive force on these forces, it is always possible to make the resultant of all forces zero. According to Saint Venant's principle, the effect of these forces on the state of stress at some distance from the ends may be neglected.

Consider, now, the problem of the bending of a circular plate of uniform thickness.

It is known that in a spherical co-ordinate system in the case of axial symmetry the biharmonic equation is of the form

$$\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi} + \frac{1}{R^2}\frac{\partial^2}{\partial\psi^2}\right)^2 \Phi = 0. \tag{9.66}$$

We first consider Laplace's equation

$$\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi} + \frac{1}{R^2}\frac{\partial^2}{\partial\psi^2}\right) \Phi = 0 \tag{9.67}$$

and try its particular solutions in the form

$$\Phi_n(R, \psi) = R^n \tilde{\Phi}_n(\psi), \tag{9.68}$$

where n is a positive integer.

Substituting (9.68) in (9.67) gives

$$\frac{d^2\tilde{\Phi}_n}{d\psi^2} + \cot\psi\frac{d\tilde{\Phi}_n}{d\psi} + n(n+1)\tilde{\Phi}_n = 0. \tag{9.69}$$

The change of the independent variable $\eta = \cos\psi$ reduces Eq. (9.69) to a Legendre equation:

$$(1-\eta^2)\frac{d^2\tilde{\Phi}_n}{d\eta^2} - 2\eta\frac{d\tilde{\Phi}_n}{d\eta} + n(n+1)\tilde{\Phi}_n = 0 \tag{9.70}$$

whose solution is sought in the form of a polynomial:

$$\tilde{\Phi}_n(\psi) = a_1\eta^n + a_2\eta^{n-2} + a_3\eta^{n-4} + \ldots + a_r\eta^{n-2r+2} + \ldots$$

Substituting this expression in Eq. (9.70), we find

$$[n(n-1)a_1+2(2n-1)a_2]\,\eta^{n-2}+\dots$$
$$\dots+\{[n(n+1)-(n-2r+2)(n-2r+3)]\,a_r+$$
$$+(n-2r+4)(n-2r+3)\,a_{r-1}\}\,\eta^{n-2r+2}=0.$$

From this

$$a_r=-\frac{(n-2r+4)(n-2r+3)}{2(r-1)(2n-2r+3)}\,a_{r-1}\quad (r=2,\ 3,\ \dots).$$

Consequently,

$$\widetilde{\Phi}_n(\psi)=a_1\left[\eta^n-\frac{n(n-1)}{2(2n-1)}\eta^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\eta^{n-4}-\dots\right].$$

We substitute this solution in (9.68). Noting that

$$\eta=\cos\psi=\frac{x_3}{R},\qquad R=\sqrt{r^2+x_3^2},$$

for $n=0,\ 1,\ 2,\ \dots$ we obtain the following solutions of Eq. (9.67):

$$\begin{aligned}
\Phi_0&=A_0,\\
\Phi_1&=A_1x_3,\\
\Phi_2&=A_2\left[x_3^2-\frac{1}{3}(r^2+x_3^2)\right],\\
\Phi_3&=A_3\left[x_3^3-\frac{3}{5}x_3(r^2+x_3^2)\right],\\
\Phi_4&=A_4\left[x_3^4-\frac{6}{7}x_3^2(r^2+x_3^2)+\frac{3}{35}(r^2+x_3^2)^2\right].
\end{aligned}\tag{9.71}$$

Here A_0, A_1, ... are unknown constant coefficients. These solutions are obviously solutions also to Eq. (9.66).

If $R^n\widetilde{\Phi}_n(\psi)$ is a solution of Eq. (9.67), it can easily be established that $R^{n+2}\widetilde{\Phi}_n(\psi)$ is a solution of Eq. (9.66). Indeed,

$$\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi}+\frac{1}{R^2}\frac{\partial^2}{\partial\psi^2}\right)R^{n+2}\,\widetilde{\Phi}_n(\psi)=$$
$$=2(2n+3)\,R^n\widetilde{\Phi}_n(\psi).$$

Substituting the last relation in Eq. (9.66), and remembering that $R^n\,\widetilde{\Phi}_n(\psi)$ is a solution of Eq. (9.67), we have

$$\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi}+\frac{1}{R^2}\frac{\partial^2}{\partial\psi^2}\right)^2R^{n+2}\,\widetilde{\Phi}_n(\psi)=$$
$$=2(2n+3)\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi}+\frac{1}{R^2}\frac{\partial^2}{\partial\psi^2}\right)R^n\widetilde{\Phi}_n(\psi)=0.$$

Consequently, on multiplying solutions (9.71) by $R^2=r^2+x_3^2$, we obtain solutions of Eq. (9.66), which are no longer solutions of

Eq. (9.67),

$$\begin{aligned}\Phi_0^* &= B_0 (r^2 + x_3^2),\\ \Phi_1^* &= B_1 x_3 (r^2 + x_3^2),\\ \Phi_2^* &= B_2 (2x_3^2 - r^2)(r^2 + x_3^2),\\ \Phi_3^* &= B_3 (2x_3^3 - 3r^2 x_3)(r^2 + x_3^2).\end{aligned} \tag{9.72}$$

By using the preceding solutions, we shall consider different cases of a symmetrically loaded circular plate (Fig. 41).

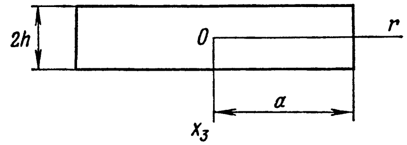

Fig. 41

(a) On the basis of (9.71) and (9.72) the stress function $\Phi(r, x_3)$ is represented as a third-degree polynomial:

$$\Phi(r, x_3) = a_3 (2x_3^3 - 3r^2 x_3) + b_3 (x_3^3 + r^2 x_3). \tag{9.73}$$

Substituting this function in formulas (9.53), we obtain

$$\begin{aligned}\sigma_{rr} &= 6a_3 + (10\nu - 2) b_3,\\ \sigma_{\varphi\varphi} &= 6a_3 + (10\nu - 2) b_3,\\ \sigma_{33} &= -12a_3 + (14 - 10\nu) b_3,\\ \sigma_{r3} &= 0.\end{aligned} \tag{9.74}$$

Thus, for the stress function (9.73) the components of the stress tensor are constant throughout the plate. The constants a_3 and b_3 can be determined if uniformly distributed $\sigma_{33} = T_{33}$ and $\sigma_{rr} = T_{rr}$ are given, respectively, on the faces and lateral surface of the plate.

(b) With the use of (9.71) and (9.72) the stress function is now represented as

$$\Phi(r, x_3) = a_4 (8x_3^4 - 24r^2 x_3^2 + 3r^4) + b_4 (2x_3^4 + r^2 x_3^2 - r^4).$$

On the basis of formulas (9.53) we obtain

$$\begin{aligned}\sigma_{rr} &= 96a_4 x_3 + 4b_4 (14\nu - 1) x_3,\\ \sigma_{33} &= -192a_4 x_3 + 8b_4 (8 - 7\nu) x_3,\\ \sigma_{r3} &= 96a_4 r - 4b_4 (8 - 7\nu) r.\end{aligned} \tag{9.74a}$$

If we assume $96a_4 - 4b_4(8 - 7\nu) = 0$, then

$$\sigma_{33} = \sigma_{r3} = 0, \quad \sigma_{rr} = 28\,(1 + \nu)\, b_4 x_3.$$

The constant b_4 can be determined if a constant value of the bending moment M_r is given on the lateral surface of the plate. Then

$$2 \int\limits_0^h \sigma_{rr} x_3\, dx_3 = M_r.$$

This condition is integral, but according to Saint Venant's principle the state of stress so found will be sufficiently accurate at points remote from the lateral surface of the plate.

From the last relation we find

$$b_4 = \frac{3M_r}{56\,(1+\nu)\,h^3}.$$

Then

$$\sigma_{rr} = \frac{3M_r}{2h^3}\, x_3, \quad \sigma_{33} = \sigma_{r3} = 0.$$

This solution represents the pure bending of the plate by moments uniformly distributed over its lateral surface.

(c) Based on (9.71) and (9.72), the stress function is taken in the form

$$\Phi\,(r,\ x_3) = \frac{1}{3}\, a_6\,(16x_3^6 - 120x_3^4r^2 + 90x_3^2r^4 - 5r^6) +$$

$$+ b_6\,(8x_3^6 - 16x_3^4r^2 - 21x_3^2r^4 + 3r^6) + a_4\,(8x_3^4 - 24r^2x_3^2 + 3r^4).$$

For this function the stresses are as follows:

$$\sigma_{rr} = a_6\,(320x_3^3 - 720r^2x_3) + b_6\,[64\,(2 + 11\nu)\,x_3^3 +$$

$$+ (504 - 48 \times 22\nu)\, r^2x_3] + 96a_4x_3,$$

$$\sigma_{33} = a_6\,(-640x_3^3 + 960r^2x_3) + b_6\,\{[-960 + 320 \times 22\,(2 - \nu)]\,x_3^3 +$$

$$+ [384 - 48 \times 22\,(2 - \nu)]\, r^2x_3\} - 192a_4x_3,$$

$$\sigma_{r3} = a_6\,(960rx_3^2 - 240r^3) + b_6\,[(-672 + 48 \times 22\nu)\,x_3^2r +$$

$$+ (432 - 12 \times 22\nu)\, r^3] + 96a_4r.$$

To the stresses σ_{33} is added a uniform tension $\sigma'_{33} = b$ in the ox_3 direction, so that the components of the stress tensor contain four constants, a_6, b_6, a_4, and b.

Let the boundary conditions be

$$\sigma_{33} = 0 \text{ when } x_3 = h,$$

$$\sigma_{33} = -p \text{ when } x_3 = -h,$$

$$\sigma_{r3} = 0 \text{ when } x_3 = \pm h,$$

where p is the intensity of uniform load.

Substituting the expressions for the stress tensor in these boundary conditions, we determine the constants a_6, b_6, a_4, and b. Consequenlty,

$$\sigma_{rr}=p\left[\frac{2+\nu}{8}\frac{x_3^3}{h^3}-\frac{3(3+\nu)}{32}\frac{r^2x_3}{h^3}-\frac{3}{8}\frac{x_3}{h}\right],$$
$$\sigma_{33}=p\left(-\frac{x_3^3}{4h^3}+\frac{3}{4}\frac{x_3}{h}-\frac{1}{2}\right), \qquad (9.75)$$
$$\sigma_{r3}=\frac{3pr}{8h^3}(h^2-x_3^2).$$

The stresses σ_{rr} on the lateral surface of the plate give bending moments M_r uniformly distributed along the contour.

To obtain the solution for a simply supported plate, to the components of the stress tensor (9.75) must be added the stresses due to pure bending, and the constant b_4 must be determined so that on the lateral surface $r=a$

$$M_r=\int_{-h}^{h}\sigma_r x_3\,dx_3=0. \qquad (9.76)$$

Then

$$\sigma_{rr}=p\left(\frac{2+\nu}{8}\frac{x_3^3}{h^3}-\frac{3(3+\nu)}{32}\frac{r^2x_3}{h^3}-\frac{3}{8}\frac{2+\nu}{5}\frac{x_3}{h}+\frac{3(3+\nu)}{32}\frac{a^2x_3}{h^3}\right). \qquad (9.77)$$

The fulfilment of condition (9.76) means that the application of pure bending eliminates the bending moments M_r on the lateral surface of the plate, the stresses σ_{rr} being equal to

$$\sigma_{rr}=\frac{2+\nu}{8}p\left(\frac{x_3^3}{h^3}-\frac{3}{5}\frac{x_3}{h}\right).$$

Noting that the resultant vector and the resultant moment of the stresses σ_{rr} are zero, it may be stated on the basis of Saint Venant's principle that the field of the stress tensor is sufficiently accurate at points remote from the lateral surface.

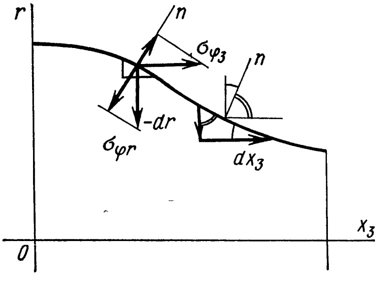

Fig. 42

Consider the torsion of a body of revolution. Let to the bases of a body of revolution (Fig. 42) be applied given forces satisfying the conditions of equilibrium of an absolutely rigid body and reducing to twisting couples. Body forces are absent, and the lateral surface of the body is free from surface forces.

This problem will be solved in terms of displacements in cylindrical co-ordinates assuming that $u_r=u_3=0$, while u_φ, because of the axial symmetry of the deformation of the body of revolution, is independent of the polar angle φ and is a function only of r and x_3. Since $u_r = u_3 = 0$ and $u_\varphi = u_\varphi(r, x_3)$, from formulas (3.29) we find

$$e_{rr} = e_{\varphi\varphi} = e_{33} = e_{3r} = 0,$$
$$e_{r\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right), \quad e_{\varphi 3} = \frac{1}{2}\frac{\partial u_\varphi}{\partial x_3}. \tag{9.78}$$

Substituting (9.78) in the formulas of Hooke's law, we have

$$\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{33} = \sigma_{3r} = 0,$$
$$\sigma_{r\varphi} = \mu\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right), \quad \sigma_{\varphi 3} = \mu\frac{\partial u_\varphi}{\partial x_3}. \tag{9.79}$$

Noting that the components $\sigma_{r\varphi}$ and $\sigma_{\varphi 3}$ are also independent of the angle φ and that body forces are absent, from Eqs. (2.30) we obtain

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{\partial \sigma_{\varphi 3}}{\partial x_3} + \frac{2\sigma_{r\varphi}}{r} = 0.$$

The last equation is rearranged in the form

$$\frac{\partial}{\partial r}(r^2\sigma_{r\varphi}) + \frac{\partial}{\partial x_3}(r^2\sigma_{\varphi 3}) = 0. \tag{9.80}$$

The solution of Eq. (9.80) is

$$\sigma_{\varphi 3} = \frac{1}{r^2}\frac{\partial \Phi}{\partial r}, \quad \sigma_{r\varphi} = -\frac{1}{r^2}\frac{\partial \Phi}{\partial x_3}. \tag{9.81}$$

Here the function $\Phi(r, x_3)$, called the stress function, is determined from the compatibility equations.

The strain compatibility conditions (3.40) for the given problem, with (9.79), take the form

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{\varphi 3})\right) - \frac{1}{r^2}\frac{\partial^2 (r^2\sigma_{r\varphi})}{\partial r\,\partial x_3} = 0;$$
$$\frac{\partial^2 \sigma_{r\varphi}}{\partial x_3^2} - r\frac{\partial^2}{\partial r\,\partial x_3}\left(\frac{\sigma_{\varphi 3}}{r}\right) = 0.$$

With (9.81), the second equation becomes

$$\frac{\partial}{\partial x_3}\left(\frac{\partial^2 \Phi}{\partial r^2} - \frac{3}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial x_3^2}\right) = 0.$$

The latter is satisfied if

$$\frac{\partial^2 \Phi}{\partial r^2} - \frac{3}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial x_3^2} = 0. \tag{9.82}$$

By direct checking it can easily be verified that under condition (9.82) the first equation is satisfied identically. Thus, the strain compatibility condition for the given problem is of the form of (9.82).

The boundary condition for the function Φ may be established by the following argument. In view of the fact that the lateral surface of the bar is free from surface forces, the sum of the projections of the shearing stresses $\sigma_{\varphi 3}$ and $\sigma_{\varphi r}$, acting at points of the boundary of an axial section, on the normal to the boundary (Fig. 42) must vanish, i.e.,

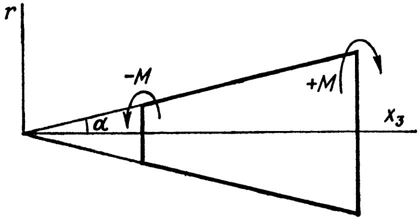

Fig. 43

$$\sigma_{\varphi r} \cos (n,\ r) + \sigma_{\varphi 3} \cos (n,\ x_3) = 0. \tag{9.83}$$

Referring to Fig. 42, we have

$$\cos (n, r) = \frac{dx_3}{dl}, \quad \cos (n, x_3) = -\frac{dr}{dl}, \tag{9.84}$$

where dl is an element of arc length of the boundary.

Substituting (9.81) and (9.84) in the boundary condition (9.83), we find

$$\frac{\partial \Phi}{\partial x_3} \frac{dx_3}{dl} + \frac{\partial \Phi}{\partial r} \frac{dr}{dl} = 0,$$

from which $\frac{d\Phi}{dl} = 0$ or $\Phi = C$.

The magnitude of the twisting moment is related to the function Φ by the equation

$$M = 2\pi \int_0^{R(x_3)} \sigma_{3\varphi} r^2\, dr = 2\pi \int_0^{R(x_3)} \frac{\partial \Phi}{\partial r}\, dr = 2\pi \{\Phi [R(x_3), x_3] - \Phi (0, x_3)\}. \tag{9.85}$$

If the body of revolution has the shape of a cone (Fig. 43), the following relation holds on its surface:

$$\frac{x_3}{\sqrt{r^2 + x_3^2}} = \cos \alpha. \tag{9.86}$$

Obviously, any function of the argument representing the left-hand side of (9.86) is a constant on the surface of the cone. We try to find the stress function in the form

$$\Phi(r, x_3) = A\left[\frac{x_3}{\sqrt{r^2+x_3^2}} + B\left(\frac{x_3}{\sqrt{r^2+x_3^2}}\right)^n\right], \tag{9.87}$$

where A, B, n are unknown constants.

It appears from the above that this function satisfies the condition $\Phi(r, x_3)$ = constant on the surface of the cone. Function (9.87) satisfies Eq. (9.82) if we assume

$$B = -\frac{1}{3}, \quad n = 3.$$

Thus,

$$\Phi(r, x_3) = A\left[\frac{x_3}{\sqrt{r^2+x_3^2}} - \frac{1}{3}\left(\frac{x_3}{\sqrt{r^2+x_3^2}}\right)^3\right].$$

The constant A is determined from (9.85):

$$A = -\frac{3M}{2\pi(2-3\cos\alpha+\cos^3\alpha)}.$$

According to formulas (9.81), the shearing stresses are

$$\sigma_{\varphi 3} = -A\frac{rx_3}{(r^2+x_3^2)^{5/2}}, \quad \sigma_{r\varphi} = -A\frac{r^2}{(r^2+x_3^2)^{5/2}}.$$

80. THERMAL STRESSES

Let us determine stresses and strains in a hollow sphere due to a steady-state temperature field when a constant temperature T_a is maintained on the inner surface of the sphere and a constant temperature T_b on the outer surface. In this problem the distribution of all required quantities is symmetrical about the centre of the sphere, i.e., all required quantities depend only on the radius r. In a spherical co-ordinate system Eq. (5.13) and the boundary conditions (5.15) become therefore

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0, \tag{9.88}$$

$$\left.\begin{aligned} T &= T_a \quad \text{when} \quad r = a, \\ T &= T_b \quad \text{when} \quad r = b. \end{aligned}\right\} \tag{9.89}$$

The solution of problem (9.88), (9.89) is

$$T = \frac{A}{r} + B, \tag{9.90}$$

where

$$A = \frac{ab}{b-a}(T_a - T_b), \quad B = \frac{ab}{b-a}\left(\frac{T_a}{b} - \frac{T_b}{a}\right).$$

Because of the symmetry of the state of stress about the centre of the sphere we have

$$\sigma_{\varphi\varphi}(r)=\sigma_{\psi\psi}(r), \quad \sigma_{r\varphi}=\sigma_{r\psi}=\sigma_{\varphi\psi}=0, \quad \sigma_r=\sigma_r(r).$$

By (2.31), the differential equation of equilibrium becomes

$$\frac{d\sigma_{rr}}{dr}+2\,\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}=0. \tag{9.91}$$

For our problem $u_\psi = u_\varphi = 0$ and $u_r = u_r(r)$; hence, from (3.32) we find

$$e_{rr}=\frac{du_r}{dr}, \quad e_{\varphi\varphi}=e_{\psi\psi}=\frac{u_r}{r}, \quad e_{\varphi r}=e_{\psi\varphi}=e_{r\psi}=0. \tag{9.92}$$

Inserting (9.92) in (4.56), we have

$$\begin{gathered}\sigma_{rr}=(\lambda+2\mu)\,\frac{du_r}{dr}+2\lambda\frac{u_r}{r}-\beta T,\\ \sigma_{\varphi\varphi}=\sigma_{\psi\psi}=\lambda\,\frac{du_r}{dr}+2\,(\lambda+\mu)\,\frac{u_r}{r}-\beta T,\\ \sigma_{\varphi r}=\sigma_{\psi\varphi}=\sigma_{r\psi}=0.\end{gathered} \tag{9.93}$$

Substituting relations (9.93) in (9.91) gives

$$\frac{d^2u_r}{dr^2}+\frac{2}{r}\,\frac{du_r}{dr}-\frac{2u_r}{r^2}=g\,\frac{dT}{dr},$$

where $g=\frac{1+\nu}{1-\nu}\,\alpha$, or

$$\frac{d}{dr}\left[\frac{1}{r^2}\,\frac{d\,(r^2u_r)}{dr}\right]=g\,\frac{dT}{dr}.$$

By integrating this equation, we find

$$u_r=\frac{g}{r^2}\int\limits_a^r Tr^2\,dr+c_1r+\frac{c_2}{r^2}. \tag{9.94}$$

Substituting the function $T = T(r)$ from (9.90) in this expression, we obtain

$$u_r=g\left[\frac{A}{2}+B\,\frac{r}{3}-\frac{A}{2}\,\frac{a^2}{r^2}-\frac{B}{3}\,\frac{a^3}{r^2}\right]+c_1r+\frac{c_2}{r^2}. \tag{9.95}$$

Inserting expression (9.95) in (9.93), we find

$$\begin{aligned}\sigma_{rr}=(\lambda+2\mu)&\left[\frac{Bg}{3}+\frac{Aga^2}{r^3}+\frac{2Bga^3}{3r^3}+c_1-\frac{2c_2}{r^3}\right]+\\ &+2\lambda\left[\frac{Ag}{2r}+\frac{Bg}{3}-\frac{Ag}{2}\,\frac{a^2}{r^3}-\frac{Bg}{3}\,\frac{a^3}{r^3}+c_1+\frac{c_2}{r^3}\right]-\beta T,\\ \sigma_{\varphi\varphi}=\sigma_{\psi\psi}=\lambda&\left[\frac{Bg}{3}+\frac{Aga^2}{r^3}+\frac{2Bga^3}{3r^3}+c_1-\frac{2c_2}{r^3}\right]+\\ &+2\,(\lambda+\mu)\left[\frac{Ag}{2r}+\frac{Bg}{3}-\frac{Ag}{2}\,\frac{a^2}{r^3}-\frac{Bg}{3}\,\frac{a^3}{r^3}+c_1+\frac{c_2}{r^3}\right]-\beta T.\end{aligned}$$

The constants c_1 and c_2 are determined from the boundary conditions $\sigma_{rr} = 0$ when $r = a$ and $r = b$.

CHAPTER X

Theory of propagation of elastic waves

81. TWO TYPES OF WAVES

The existence of two types of waves in a homogeneous isotropic medium was first proved by S. D. Poisson; one type of wave is known as compression-dilatation waves, the other as shear waves. Poisson showed that they are characterized by different velocities of wave front propagation, and also by the fact that compression-dilatation waves involve no rotation of particles, while shear waves are not accompanied by a change of volume.

We proceed to the proof of the existence of two types of waves. Let us consider an infinite medium. The mass forces F acting on this medium and the displacement field u are represented as

$$F = \nabla\Phi + \operatorname{rot} \boldsymbol{\Psi}, \tag{10.1}$$

$$u = \nabla\varphi + \operatorname{rot} \psi. \tag{10.2}$$

Here Φ and φ are scalar functions of the co-ordinates (x_1, x_2, x_3) and the time t, and $\boldsymbol{\Psi}$ and ψ are vector functions of the co-ordinates and the time t.

It follows from (10.2) that

$$\operatorname{div} u = \Delta\varphi. \qquad (10.3$$

Substituting expressions (10.1) and (10.2) in the equation of motion of an elastic medium (5.5), taking into account (10.3), and interchanging the order of the differential operators, we obtain

$$\nabla\left[c_1^2\Delta\varphi - \frac{\partial^2\varphi}{\partial t^2} + \Phi\right] + \operatorname{rot}\left[c_2^2\Delta\psi - \frac{\partial^2\psi}{\partial t^2} + \boldsymbol{\Psi}\right] = 0, \tag{10.4}$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}. \tag{10.5}$$

It is easy to see that (10.4) is satisfied if we assume

$$c_1^2\Delta\varphi - \frac{\partial^2\varphi}{\partial t^2} = -\Phi, \tag{10.6}$$

$$c_2^2\Delta\psi - \frac{\partial^2\psi}{\partial t^2} = -\boldsymbol{\Psi}. \tag{10.7}$$

Thus, it is proved that the vector field u defined by (10.2) is the solution of Eq. (5.5) if the functions φ and ψ satisfy (10.6) and (10.7); the function φ is called the longitudinal potential, and ψ is the transverse potential.

The question now arises as to whether these equations have solutions that cannot be represented in the form indicated above. It may be shown that there are no such solutions. We shall mention some important consequences.

(a) Let $\Psi \equiv 0$ and let the initial conditions be $\psi = 0$ when $t = t_0$. The resulting equation for the determination of ψ is then the homogeneous equation (10.7) with zero initial conditions. This means that ψ is always zero; it follows from Eq. (10.2) that $u = \nabla\varphi$ and rot $u = 0$.

This shows that a wave described by the function φ involves no rotation of the particles of the medium, i.e., each of them has a motion of translation. Such waves are therefore called longitudinal. It must be emphasized once again that if $\Psi \equiv 0$ and if at a certain instant the wave field is longitudinal in nature, it always remains so, i.e., longitudinal waves propagating in an isotropic homogeneous infinite medium do not generate transverse waves.

Equation (10.6) describing longitudinal waves is a non-homogeneous wave equation. It is known that if the function Φ and the initial conditions in a finite part of space are different from zero, then the surface separating the disturbed from the undisturbed region (the wave front) is propagated in the direction of its normal towards the undisturbed region with a velocity c_1.

(b) Let now $\Phi \equiv 0$ and let the initial conditions be $\varphi = 0$ when $t = t_0$. Then $\varphi \equiv 0$ and $u = \text{rot}\,\psi$. In this field the dilatation is zero. Indeed, div $u =$ div rot $\psi = 0$.

Waves possessing this property are called transverse or shear waves. Transverse waves propagating in an infinite medium do not generate longitudinal waves. The velocity of propagation of the transverse wave front is c_2.

In a homogeneous medium with boundary the longitudinal and transverse waves travel independently only until the front intersects the boundary. Waves (reflected waves, as they are called) of both types are then formed for it is usually impossible to satisfy the system of boundary conditions by introducing a reflected wave of any one type.

Consider several examples.

1. *Plane longitudinal wave.* Suppose that body forces are absent, the transverse potential ψ is identically zero, and the longitudinal potential φ depends only on x_1 and t. Equation (10.6) then transforms into the equation of vibration of a string.

$$c_1^2 \frac{\partial^2 \varphi}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial t^2} = 0$$

and has a solution of the form

$$\varphi = f(x_1 - c_1 t) + g(x_1 + c_1 t), \tag{10.8}$$

where f and g are arbitrary twice differentiable functions.

The first term in (10.8) represents a wave of constant shape moving with the velocity c_1 in the positive direction of the x_1 axis, and the second term represents a wave of constant shape moving in the opposite direction.

The displacement corresponding to solution (10.8) is, by (10.2),

$$u_1 = \frac{\partial \varphi}{\partial x_1} = f'(x_1 - c_1 t) + g'(x_1 + c_1 t). \tag{10.9}$$

Expression (10.9) shows that for a fixed t the wave field on each plane perpendicular to the x_1 axis does not change from point to point and is parallel to the x_1 axis. If the direction of propagation of a plane wave does not coincide with the x_1 axis, the displacement field is described by more complicated formulas, although the physical picture remains the same. Let us derive the corresponding formulas.

Let the direction of propagation of plane longitudinal waves $\boldsymbol{n}$ make with the co-ordinate axes angles whose cosines are n_k. Denote by l the distance measured along a straight line parallel to the direction $\boldsymbol{n}$. For simplicity, we consider a wave travelling in one direction. Substituting for l its expression $l = x_k n_k$ $(k = 1, 2, 3)$, we obtain

$$\varphi = f(l - c_1 t) = f(x_k n_k - c_1 t).$$

From (10.2), the components of the displacement vector are obtained as

$$u_\nu = n_\nu f'(x_k n_k - c_1 t) \quad (\nu = 1, 2, 3).$$

2. *Spherical longitudinal wave.* Consider the case when the longitudinal potential φ in a spherical co-ordinate system depends only on the radius r and the time t. The transverse potential ψ is again identically zero. Body forces are absent.

In this case Eq. (10.6) in spherical co-ordinates becomes

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{2}{r}\frac{\partial \varphi}{\partial r} - \frac{1}{c_1^2}\frac{\partial^2 \varphi}{\partial t^2} = 0$$

or

$$\frac{\partial^2}{\partial r^2}(r\varphi) - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}(r\varphi) = 0.$$

The solution of this equation is

$$r\varphi = f(r - c_1 t) + g(r + c_1 t);$$

hence,

$$\varphi = \frac{1}{r} f(r - c_1 t) + \frac{1}{r} g(r + c_1 t). \tag{10.10}$$

The first term in (10.10) represents a wave diverging from the centre, and the second term represents a wave moving towards the centre.

Consider a wave diverging from the centre. Since the longitudinal potential φ depends only on r and t, the only non-vanishing projection of the vector $\boldsymbol{u}$ in spherical coordinates (r, ψ, φ) is

$$u_r = \frac{\partial \varphi}{\partial r} = \frac{1}{r} f'(r - c_1 t) - \frac{1}{r^2} f(r - c_1 t). \qquad (10.11)$$

Expression (10.11) indicates that the displacement u_r is directed strictly along the radius and does not change from point to point if the points lie on the same sphere (for a fixed time t).

It is important to emphasize that as r tends to zero, u_r tends to infinity, and so do the strains and stresses. In general, Lamé's equations are unsuitable to describe a medium undergoing large deformations. But formally these equations admit such solutions and they are suitable, and convenient, for describing real processes when r is bounded from below. Suppose, for example, that an elastic wave is produced by a uniform pressure applied to the surface of a spherical cavity of radius r_0. Formula (10.11) then describes the solution in the region $r \geqslant r_0$, and the singularity as $r \to 0$ is found to be outside the region in which the solution is sought. In this example the function f appearing in formula (10.11) is easily determined from the given pressure $p = p(r_0, t)$ on the cavity.

Thus, solution (10.11) has a singularity at $r = 0$. This singularity is called the centre of dilatation. Note that, in contrast to a plane wave which does not change its shape during propagation, a spherical wave does change its shape. Indeed, the coefficients $\frac{1}{r}$ and $\frac{1}{r^2}$ in formula (10.11) show that the wave amplitudes change with r.

3. *Plane transverse wave.* Suppose that body forces are again absent; the longitudinal potential $\varphi \equiv 0$, and the transverse potential ψ has only one non-vanishing component ψ_3, which depends on x_1 and t alone. From (10.7) we obtain

$$c_2^2 \frac{\partial^2 \psi_3}{\partial x_1^2} - \frac{\partial^2 \psi_3}{\partial t^2} = 0.$$

Hence

$$\psi_3 = f(x_1 - c_2 t) + g(x_1 + c_2 t).$$

For simplicity, we consider only a wave travelling in the positive direction of the x_1 axis. The projections of the displacement vector are given by the formulas

$$u_1 = u_3 = 0, \quad u_2 = -f'(x_1 - c_2 t).$$

Here the direction of the x_1 axis is the direction of propagation of the wave. In contrast to a plane longitudinal wave, however, its velocity of propagation is c_2, and the direction of its displacement does not coincide with the direction of wave propagation, but is perpendicular to it (in the present case the displacement is directed along the x_2 axis).

It can easily be verified that the dilatation in this wave, as in the general case of a transverse wave, is zero:

$$\operatorname{div} u = \frac{\partial u_2}{\partial x_2} = 0.$$

According to formulas (3.27), the components of the tensor of rotation of particles are obtained as

$$\omega_1 = \omega_2 = 0, \quad \omega_3 = \frac{1}{2} f''(x_1 - c_2 t),$$

i.e., the particles rotate along an axis parallel to the x_3 axis.

82. RAYLEIGH SURFACE WAVES

Consider an elastic half-space. Let the origin of coordinates be placed on its surface, with the x_1 axis directed along the boundary and the x_2 axis into the medium (Fig. 44). It is assumed that body forces are absent. We seek a solution of Eqs. (10.6) and (10.7) that is independent of x_3 (plane strain), varies in time according to a sine law, dies off with depth, and satisfies the conditions $T_{21} = T_{22} = 0$ on the boundary $x_3 = 0$. When $x_2 = 0$, we have

$$\sigma_{22} = \sigma_{21} = 0. \qquad (10.12)$$

This is a problem of free vibrations of a half-space.

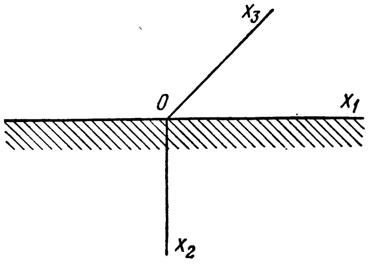

Fig. 44

The solution is sought in the form:

$$\varphi = Ae^{-\alpha x_2 + iq(x_1 - ct)} \quad (\alpha > 0), \qquad (10.13)$$

$$\psi_3 = Be^{-\beta x_2 + iq(x_1 - ct)} \quad (\beta > 0),$$

$$\psi_1 = \psi_2 = 0.$$

Here q is a given frequency. The constants α, β, c (c is the phase velocity), A, B must be chosen so that (10.13) will satisfy Eqs. (10.6), (10.7) and the boundary conditions (10.12).

Substituting (10.13) in (10.6) and (10.7), we obtain, after simple manipulation,

$$\alpha = q\sqrt{1 - \frac{c^2}{c_1^2}}, \qquad (10.14)$$

$$\beta = q\sqrt{1 - \frac{c^2}{c_2^2}}.$$

On the basis of solution (10.13) and formula (10.2) we find

$$u_1 = (iqAe^{-\alpha x_2} - \beta Be^{-\beta x_2})\, e^{iq(x_1-ct)},$$
$$u_2 = (-\alpha Ae^{-\alpha x_2} - iqBe^{-\beta x_2})\, e^{iq(x_1-ct)}.$$

Consequently, the displacement vector $\boldsymbol{u}$ is in planes perpendicular to the x_3 axis.

The displacements on the boundary $x_2 = 0$ are

$$u_1 = (iqA - \beta B)\, e^{-iq(x_1-ct)}, \quad u_2 = -(\alpha A + iqB)\, e^{iq(x_1-ct)}. \tag{10.15}$$

By using the formulas for the displacement and Hooke's law, it is easy to obtain expressions for the components of the stress tensor on the boundary:

$$\sigma_{22} = \mu q^2 \left\{ \left(2 - \frac{c^2}{c_2^2}\right) A + 2i \sqrt{1 - \frac{c^2}{c_2^2}}\, B \right\} e^{iq(x_1-ct)},$$
$$\sigma_{12} = \mu q^2 \left\{ -2i \sqrt{1 - \frac{c^2}{c_1^2}}\, A + \left(2 - \frac{c^2}{c_2^2}\right) B \right\} e^{iq(x_1-ct)}.$$

In order to satisfy the boundary conditions (10.12), it is necessary to put

$$\begin{aligned} \left(2 - \frac{c^2}{c_2^2}\right) A + 2i \sqrt{1 - \frac{c^2}{c_2^2}}\, B &= 0, \\ -2i \sqrt{1 - \frac{c^2}{c_1^2}}\, A + \left(2 - \frac{c^2}{c_2^2}\right) B &= 0. \end{aligned} \tag{10.16}$$

We have obtained a linear homogeneous system of equations in A and B. For A and B to be different from zero, the determinant R of this system must be set equal to zero:

$$R = \left(2 - \frac{c^2}{c_2^2}\right)^2 - 4 \sqrt{\left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{c_2^2}\right)} = 0 \tag{10.17}$$

or

$$R \equiv (2-k)^2 - 4\sqrt{(1-k)(1-\gamma k)} = 0,$$

where

$$k = \frac{c^2}{c_2^2}, \quad \gamma = \frac{c_2^2}{c_1^2} < 1.$$

This equation determines the phase velocity c; it is important to emphasize that the latter is independent of the frequency q, but depends only on the ratio c_2/c_1.

Let us show that $c^2 < c_2^2$. Indeed, putting $c = c_2$, we obtain $R = 1$. On the other hand, when $c = 0$, we have $R = 0$ and $R' = -2 \times$ $\times (1 - \gamma) < 0$. It follows from this that Eq. (10.17) has the root

$k = \frac{c^2}{c_2^2} < 1$ for all values of c_2/c_1 (Fig. 45). It may be shown that there are no other roots on the interval [0, 1] (the root $c = 0$ corresponding to the zero solution not being considered). In particular, when $\lambda = \mu$, i.e., when $c_2/c_1 = \frac{1}{\sqrt{3}}$,

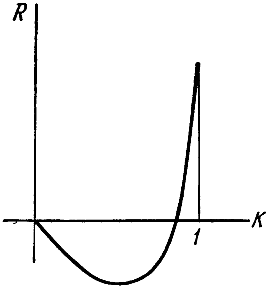

Fig. 45

$$c = \sqrt{2\left(1 - \frac{1}{\sqrt{3}}\right)}\, c_2.$$

From (10.16) we obtain

$$A = -2i\sqrt{1 - \frac{c^2}{c_2^2}}\,\frac{D}{q}, \quad B = \left(2 - \frac{c^2}{c_2^2}\right)\frac{D}{q},$$

where D is an arbitrary constant. Then

$$u_1 = D\sqrt{1 - \frac{c^2}{c_2^2}}\left[2e^{-\alpha x_2} - \left(2 - \frac{c^2}{c_2^2}\right)e^{-\beta x_2}\right]e^{iq(x_1 - ct)},$$

$$u_2 = iD\left[2\sqrt{\left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{c_2^2}\right)}\,e^{-\alpha x_2} - \left(2 - \frac{c^2}{c_2^2}\right)e^{-\beta x_2}\right]e^{iq(x_1 - ct)}.$$

We have constructed the solution in complex form, but since the equations and the boundary conditions of the problem are linear, its solution is given by both the real and the imaginary part of the resulting expressions; for example,

$$u_1 = D\sqrt{1 - \frac{c^2}{c_2^2}}\left[2e^{-\alpha x_2} - \left(2 - \frac{c^2}{c_2^2}\right)e^{-\beta x_2}\right]\cos q(x_1 - ct), \quad (10.18)$$

$$u_2 = -D\left[2\sqrt{\left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{c_2^2}\right)}\,e^{-\alpha x_2} - \right.$$

$$\left. - \left(2 - \frac{c^2}{c_2^2}\right)e^{-\beta x_2}\right]\sin q(x_1 - ct).$$

Since the coefficients α and β [formulas (10.14)], characterizing the attenuation with depth, grow with increasing frequency q, we deduce from (10.18) that the longer the wave, the greater is the depth at which it has an effect.

When $x_2 = 0$, from (10.18) we obtain

$$u_1 = D\sqrt{1 - \frac{c^2}{c_2^2}}\,\frac{c^2}{c_2^2}\cos q(x_1 - ct),$$

$$u_2 = -D\left[2\sqrt{\left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{c_2^2}\right)} - \left(2 - \frac{c^2}{c_2^2}\right)\right]\sin q(x_1 - ct). \quad (10.19)$$

It follows from this that the points of the surface move in ellipses.

The waves considered above were first studied by Rayleigh (J. W. Strutt). They are observed far from the disturbance source.

Since the energy carried by these waves is concentrated at the surface and is dissipated over the surface, its dissipation is slower than in waves where the energy is dissipated over the volume of the disturbed region. During earthquakes, therefore, for an observer remote from the epicentre the Rayleigh waves represent the greatest danger.

83. LOVE WAVES

Consider an elastic layer of constant thickness H with elastic constants λ, μ and density ρ, resting on an elastic half-space with

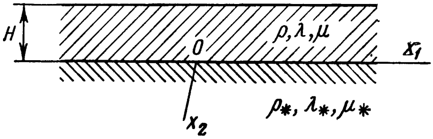

Fig. 46

parameters λ_*, μ_*, ρ_*. Assume that the velocity of transverse waves in the layer c_2 is less than the corresponding velocity c_{*2} in the half-space:

$$c_2 < c_{*2}. \tag{10.20}$$

Let the x_1 axis be taken along the interface, with the x_2 axis directed into the half-space (Fig. 46).

Let the boundary of the layer $x_2 = -H$ be free from load, i.e.,

$T_{21} = T_{22} = T_{23} = 0$. When $x_2 = -H$,

$$\sigma_{22} = \sigma_{12} = \sigma_{23} = 0, \tag{10.21}$$

and at the interface

$$u_1 = u_1^*, \quad u_2 = u_2^*, \quad u_3 = u_3^*, \tag{10.22}$$

$$\sigma_{22} = \sigma_{22}^*, \quad \sigma_{12} = \sigma_{12}^*, \quad \sigma_{23} = \sigma_{23}^* \tag{10.23}$$

(starred quantities refer to the half-space). In addition we require that as x_2 tends to infinity the displacements should tend to zero. We shall try to find solutions of Eq. (5.5) for the layer and the half-space such that the only non-zero components are u_3 and u_3^* and these are independent of x_3. Such a wave, if it exists, is a transverse one since div $\boldsymbol{u} = 0$.

From Eq. (5.5) (without considering body forces) we obtain

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} = \frac{1}{c_2^2}\frac{\partial^2 u_3}{\partial t^2}, \quad \frac{\partial^2 u_3^*}{\partial x_1^2} + \frac{\partial^2 u_3^*}{\partial x_2^2} = \frac{1}{c_{*2}^2}\frac{\partial^2 u_3^*}{\partial t^2}. \tag{10.24}$$

In view of the above assumptions regarding the displacements the first pairs of conditions (10.21), (10.22), and (10.23) are automatically satisfied, and the last give the following results: when $x_2 = -H$,

$$\frac{\partial u_3}{\partial x_2} = 0; \tag{10.25}$$

when $x_2 = 0$,

$$u_3 = u_3^*, \quad \mu \frac{\partial u_3}{\partial x_2} = \mu_* \frac{\partial u_3^*}{\partial x_2}. \tag{10.26}$$

We shall seek solutions whose dependence on x_1 and t is described by a sine law, i.e.,

$$u_3 = f(x_2)\, e^{iq(x_1 - ct)}, \quad u_3^* = f_*(x_2)\, e^{iq(x_1 - ct)}. \tag{10.27}$$

Here q is a given frequency, c is unknown phase velocity about which we assume $c_2 < c < c_{*2}$, this being consistent with (10.20).

Substituting (10.27) in Eqs. (10.24), we obtain

$$\begin{gathered} f'' + q^2\alpha^2 f = 0 \left(\alpha = \sqrt{\frac{c^2}{c_2^2} - 1}\right), \\ f_*'' - q^2\beta^2 f_* = 0 \left(\beta = \sqrt{1 - \frac{c^2}{c_{*2}^2}}\right), \end{gathered} \tag{10.28}$$

from which

$$\begin{gathered} f(x_2) = A \sin(\alpha q x_2) + B \cos(\alpha q x_2), \\ f_*(x_2) = Ce^{-\beta q x_2} + C_1 e^{\beta q x_2}. \end{gathered} \tag{10.29}$$

For the solution $f_*(x_2)$ to be bounded we must put $C_1 = 0$, then

$$f_*(x_2) = Ce^{-\beta q x_2}. \tag{10.30}$$

It follows from the boundary conditions (10.26) that

$$B = C, \quad A = -\frac{\mu_* \beta}{\mu \alpha} C. \tag{10.31}$$

Substituting (10.27) in (10.25), and using (10.29), we obtain

$$A \cos(\alpha q H) + B \sin(\alpha q H) = 0$$

or, with (10.31),

$$\tan(\alpha q H) = \frac{\mu_* \beta}{\mu \alpha}. \tag{10.32}$$

Since α and β are expressed in terms of c, c_2, c_{*2} by means of formulas (10.28), it follows that (10.32) is an equation for the determination of the ratio c/c_2 as a function of the parameters qH, c_2/c_{*2}, μ_*/μ.

Let us show that the roots of Eq. (10.32) exist. We assume the parameter $\xi = qH$ to be unknown, and the remaining parameters to be given. As $\xi = qH$ varies from zero to $\frac{\pi}{2\alpha}$, $\tan(\alpha qH)$ varies from zero to infinity, and since the tangent is a continuous function, there is a value $\xi = \xi_0$ for which (10.32) is satisfied. This proves the existence of the root of Eq. (10.32).

We write down the final formulas for displacements:

$$u_3 = C\left[\cos(\alpha q x_2) - \frac{\mu_* \beta}{\mu\alpha}\sin(\alpha q x_2)\right] e^{iq(x_1 - ct)},$$

$$u_3^* = Ce^{-q\beta x_2 + iq(x_1 - ct)}.$$

The solution obtained represents a wave running in the direction of the x_1 axis with the velocity c. The displacements in the wave are in a plane perpendicular to the direction of propagation and parallel to the boundaries of the layer. It is essential to note that their phase velocity depends on frequency q (see 10.32), i.e., these waves have dispersion.

These waves were first discovered by A. E. Love and therefore they are called after his name. Love waves, while differing from Rayleigh waves by the presence of dispersion, by their purely transverse character, etc., have many features in common with them. As Rayleigh waves, they are usually observed during earthquakes at considerable distances from the epicentre. As in Rayleigh waves, the energy in Love waves is concentrated near the interface, and hence they are attenuated more slowly than other waves.

CHAPTER XI

Theory of thin plates

84. DIFFERENTIAL EQUATION FOR BENDING OF THIN PLATES

A body having the middle surface in the form of a plane and whose thickness is sufficiently small compared with its other two dimensions is called a thin plate. Plates find wide application in engineering; as typical examples we may mention concrete and reinforced concrete plates used in structures, for ship hulls. A plane dividing the thickness of the plate in half is called its middle plane. We choose the axes of co-ordinates x_1 and x_2 in the middle plane, and the x_3 axis perpendicular to it.

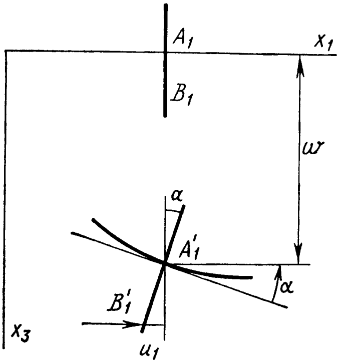

Fig. 47

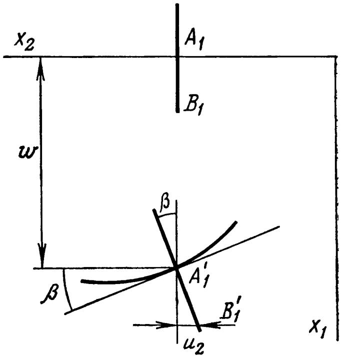

Fig. 48

If the deflection of the middle plane of a plate is small compared with the plate thickness, the following assumptions apply: (1) a normal to the middle plane before bending transforms into a normal to the middle plane after bending; (2) the component σ_{33} of the stress tensor is small compared with the other components of the stress tensor; (3) during bending the middle plane of the plate does not deform.

Let the deflection of the middle plane be denoted by w, and the displacements parallel to the x_1 and x_2 axes by u_1 and u_2, respectively.

Consider sections of the plate parallel to the planes x_1x_3 and x_2x_3, as shown in Figs. 47 and 48, respectively. Referring to these figures, and remembering the first assumption, for displacements of a point B lying on a normal to the middle plane of the plate we have

$$u_1 = -x_3 \tan \alpha, \; u_2 = -x_3 \tan \beta.$$

Since the deflection is considered to be small, it follows that

$$\alpha = \tan \alpha = \frac{\partial w}{\partial x_1}, \quad \beta = \tan \beta = \frac{\partial w}{\partial x_2}.$$

Taking into account the last relations, we find

$$u_1 = -x_3 \frac{\partial w}{\partial x_1}, \quad u_2 = -x_3 \frac{\partial w}{\partial x_2}. \tag{11.1}$$

From formulas (3.26), with (11.1), we find

$$e_{11} = -x_3 \frac{\partial^2 w}{\partial x_1^2}, \quad e_{22} = -x_3 \frac{\partial^2 w}{\partial x_2^2}, \quad e_{12} = -x_3 \frac{\partial^2 w}{\partial x_1 \, \partial x_2}. \tag{11.2}$$

By virtue of the assumption (1) we have

$$e_{13} = e_{23} = 0. \tag{11.3}$$

On the basis of the assumption (2) we put $\sigma_{33} = 0$; by formulas (11.2), Hooke's law becomes

$$\begin{aligned} \sigma_{11} &= -\frac{Ex_3}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x_1^2} + \nu \frac{\partial^2 w}{\partial x_2^2}\right), \\ \sigma_{22} &= -\frac{Ex_3}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x_2^2} + \nu \frac{\partial^2 w}{\partial x_1^2}\right), \\ \sigma_{12} &= -\frac{Ex_3}{1+\nu} \frac{\partial^2 w}{\partial x_1 \, \partial x_2}. \end{aligned} \tag{11.4}$$

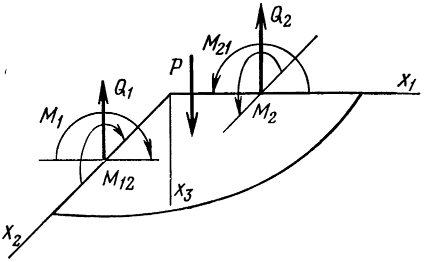

Fig. 49

Denote by M_1, M_2 the bending moments, by $M_{12} = M_{21}$ the twisting moments, and by Q_1, Q_2 the shearing forces per unit length of sections parallel to the planes x_1x_3 and x_2x_3 (Fig. 49), i.e.,

$$\begin{gathered} M_1 = \int_{-h/2}^{h/2} \sigma_{11} x_3 \, dx_3, \qquad M_2 = \int_{-h/2}^{h/2} \sigma_{22} x_3 \, dx_3, \\ M_{12} = M_{21} = \int_{-h/2}^{h/2} \sigma_{21} x_3 \, dx_3, \\ Q_1 = \int_{-h/2}^{h/2} \sigma_{31} \, dx_3, \qquad Q_2 = \int_{-h/2}^{h/2} \sigma_{32} \, dx_3. \end{gathered} \tag{11.5}$$

In spite of the fact that according to (11.3) it is necessary to put $\sigma_{31} = 2\mu e_{31} = 0$, $\sigma_{32} = 2\mu e_{32} = 0$, in setting up the equations of equilibrium we must take into account the resultant forces (shearing forces) Q_1 and Q_2 due to the shearing stresses σ_{31} and σ_{23} as quantities of the same order of magnitude as the intensity of transverse force p and the moments M_1, M_2, and M_{12}.

Substituting the expressions for σ_{11}, σ_{22}, and σ_{12} in the first three relations of (11.5), we find, for a homogeneous plate,

$$\begin{aligned} M_1 &= -D\left(\frac{\partial^2 w}{\partial x_1^2} + \nu\,\frac{\partial^2 w}{\partial x_2^2}\right), \\ M_2 &= -D\left(\frac{\partial^2 w}{\partial x_2^2} + \nu\,\frac{\partial^2 w}{\partial x_1^2}\right), \\ M_{12} &= D(1-\nu)\,\frac{\partial^2 w}{\partial x_1\,\partial x_2}, \end{aligned} \tag{11.6}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate.

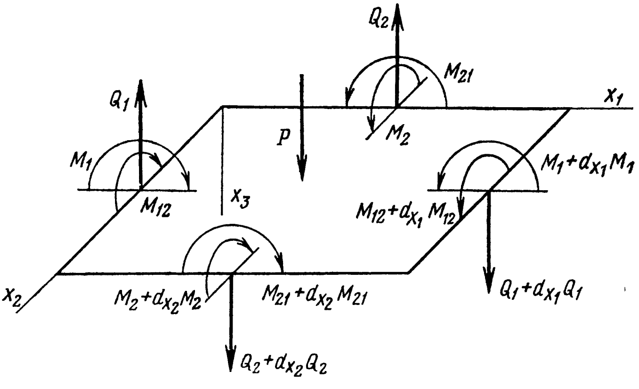

Fig. 50

Consider an element cut from the plate by two pairs of planes parallel to the co-ordinate planes x_1x_3 and x_2x_3 (Fig. 50). For equilibrium of this element it is necessary that the sum of the forces acting on this element and the sum of their moments about the x_1 and x_2 axes separately should be equal to zero. Disregarding body forces, and neglecting small quantities of the third order, we have

$$\begin{aligned} d_{x_1}Q_1\,dx_2 + d_{x_2}Q_2\,dx_1 + p\,dx_1\,dx_2 &= 0, \\ d_{x_1}M_{12}\,dx_2 - d_{x_2}M_2\,dx_1 + Q_2\,dx_1\,dx_2 &= 0, \\ d_{x_2}M_{21}\,dx_1 - d_{x_1}M_1\,dx_2 + Q_1\,dx_2\,dx_1 &= 0. \end{aligned}$$

Here d_{x_k} is the partial differential of a function that follows it with respect to the x_k co-ordinate. After some manipulation we find

$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}+p=0, \tag{11.7}$$

$$\frac{\partial M_{12}}{\partial x_1}-\frac{\partial M_2}{\partial x_2}+Q_2=0, \tag{11.8}$$

$$\frac{\partial M_{21}}{\partial x_2}-\frac{\partial M_1}{\partial x_1}+Q_1=0. \tag{11.9}$$

Inserting relations (11.6) in (11.8) and (11.9), we have, for a plate of constant thickness,

$$\left.\begin{aligned} Q_1&=-D\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\right),\\ Q_2&=-D\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\right).\end{aligned}\right. \tag{11.10}$$

Substituting the expressions for Q_1 and Q_2 in Eq. (11.7), we obtain

$$\frac{\partial^4 w}{\partial x_1^4}+2\frac{\partial^4 w}{\partial x_1^2\,\partial x_2^2}+\frac{\partial^4 w}{\partial x_2^4}=\frac{p}{D}. \tag{11.11}$$

This equation was first derived by Sophie Germain.

Thus, the problem of a plate bent by a transverse force p is reduced to the integration of Eq. (11.11).

85. BOUNDARY CONDITIONS

Let us establish the boundary conditions for a rectangular plate corresponding to several modes of fixing its edges; the x_1 and x_2 axes are directed parallel to the edges of the plate.

(a) *Clamped edge.* If the edge $x_1 = 0$ of the plate is clamped, the deflection at the points of this edge is zero and the plane tangential to the deflected middle surface coincides with the middle plane of the plate before bending:

$$w|_{x_1=0}=0, \qquad \left.\frac{\partial w}{\partial x_1}\right|_{x_1=0}=0. \tag{11.12}$$

(b) *Simply supported edge.* If the edge $x_1 = 0$ of the plate is supported and is free to rotate, the deflection and the bending moment at this edge must be zero:

$$w|_{x_1=0}=0, \qquad \frac{\partial^2 w}{\partial x_1^2}+\nu\frac{\partial^2 w}{\partial x_2^2}=0.$$

Since along the edge $x_1 = 0$ we have $w = 0$, it follows that

$$w|_{x_1=0}=\left.\frac{\partial w}{\partial x_2}\right|_{x_1=0}=\left.\frac{\partial^2 w}{\partial x_2^2}\right|_{x_1=0}=0.$$

Consequently, the boundary conditions for a simply supported edge are

$$w|_{x_1=0}=0, \qquad \left.\frac{\partial^2 w}{\partial x_1^2}\right|_{x_1=0}=0. \tag{11.13}$$

(c) *Free edge.* If the edge $x_1 = 0$ is free, it is necessary, at first glance, to require that the bending moment M_1, the twisting moment M_{12}, and the shearing force Q_1 along it should be zero

$$M_1|_{x_1=0}=0, \qquad M_{12}|_{x_1=0}=0, \qquad Q_1|_{x_1=0}=0. \tag{11.14}$$

Thus, in this case there are three boundary conditions whereas there were two of them in the other cases. Conditions (11.14) were obtained by S. D. Poisson. Later G. Kirchhoff showed that two boundary conditions sufficed to determine completely the deflection w satisfying Eq. (11.11) because two Poisson's conditions relating to the twisting moment M_{12} and the shearing force Q_1 may be combined into one boundary condition. Consequently, Poisson's system of boundary conditions (11.14) for Sophie Germain's equation (11.11) is overdetermined.

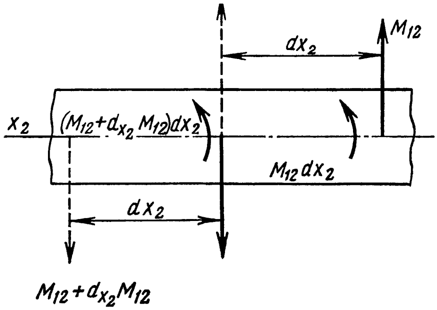

Fig. 51

Consider two adjacent elements of length dx_2 at the edge $x_1 = 0$ (Fig. 51). The twisting moment per the element of length dx_2 is $M_{12}dx_2$; it may be replaced by two shearing forces equal to M_{12} and acting at a distance dx_2 apart; in Fig. 51 these forces are shown by solid vectors. For the next element dx_2 the twisting moment $(M_{12} + d_{x_2}M_{12})\,dx_2$ may also be replaced by two shearing forces, $M_{12} + d_{x_2}M_{12}$; they are shown by dashed vectors. Thus, we find that the distribution of twisting moments M_{12} is statically equivalent to the distribution of shearing forces of intensity $Q_1' = -\frac{\partial M_{12}}{\partial x_2}$. On the basis of Saint Venant's principle this replacement will have an effect on the state of stress in the immediate vicinity of the edge, but the state of stress in the remaining part of the plate will remain unchanged.

Consequently, instead of the last two in the boundary conditions (11.14) for a free edge of a plate we obtain one condition:

$$\left(Q_1-\frac{\partial M_{12}}{\partial x_2}\right)_{x_1=0}=0. \tag{11.15}$$

On the basis of relations (11.6) and (11.10) for a free edge the boundary conditions (11.15) and $M_1|_{x_1=0} = 0$ may be expressed as

$$\left.\frac{\partial^3 w}{\partial x_1^3}+(2-\nu)\frac{\partial^3 w}{\partial x_1\,\partial x_2^2}\right|_{x_1=0}=0,\qquad\left.\frac{\partial^2 w}{\partial x_1^2}+\nu\frac{\partial^2 w}{\partial x_2^2}\right|_{x_1=0}=0. \tag{11.16}$$

In the case of a plate with a curved edge the co-ordinate axes at a point of the edge are taken to coincide with the normal ***n*** and the tangent τ, as shown in Figs. 52 and 53. The relations between M_n, $M_{n\tau}$, Q_n and M_1, M_{12}, Q_1, Q_2 are determined from the conditions for the equilibrium of an element of the plate, such as represented in Figs. 52 and 53:

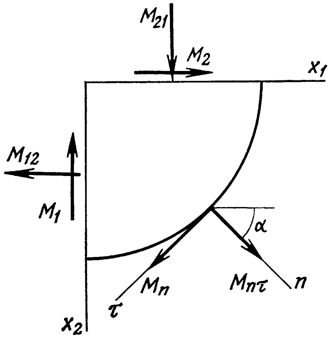

Fig. 52

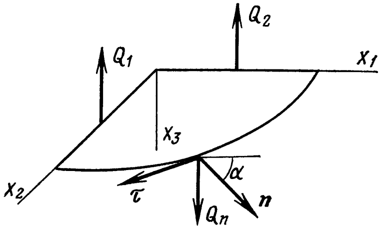

Fig. 53

$$M_n = M_1\cos^2\alpha + M_2\sin^2\alpha - M_{12}\sin 2\alpha,$$
$$M_{n\tau} = M_{12}\cos^2 2\alpha + \frac{M_1-M_2}{2}\sin 2\alpha, \tag{11.17}$$
$$Q_n = Q_1\cos\alpha + Q_2\sin\alpha.$$

When the curved edge of the plate is clamped,

$$w=0,\quad \frac{\partial w}{\partial n}=0; \tag{11.18}$$

in the case of the simply supported edge

$$w=0,\quad M_n=0.$$

If the edge of the plate is free, then

$$M_n=0,\quad Q_n-\frac{\partial M_{n\tau}}{\partial\tau}=0, \tag{11.19}$$

where the term $-\frac{\partial M_{n\tau}}{\partial\tau}$ is obtained similarly to Fig. 51.

86. BENDING EQUATION FOR A PLATE REFERRED TO POLAR CO-ORDINATES

In studying the bending of a circular plate it is advantageous to use a polar system of co-ordinates (r, φ). In this co-ordinate system, on the basis of the formulas expressing the relation between polar and Cartesian co-ordinates

$$r^2 = x_1^2 + x_2^2, \qquad \varphi = \operatorname{arc\,tan} \frac{x_2}{x_1}, \tag{11.20}$$

the harmonic operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{11.21}$$

Consequently, the bending equation for a plate (11.11) in a polar co-ordinate system is written as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) w = \frac{p}{D}. \tag{11.22}$$

If the load p is distributed symmetrically about the centre of the plate, the deflection w depends only on the polar radius. In this case Eq. (11.22) becomes

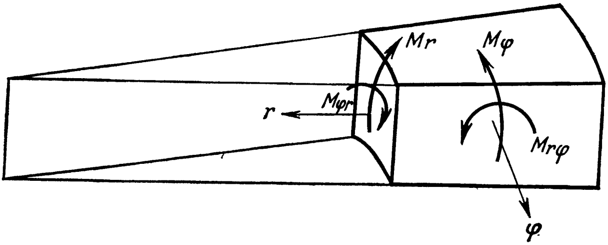

Fig. 54

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) w = \frac{p}{D}.$$

Let the bending moments acting at sections with normals r and φ be denoted by M_r and M_φ, respectively, and the twisting moment by $M_{r\varphi}$. These moments, as usual, are calculated per unit length. Suppose that the ox_1 axis coincides with the polar radius r; then the moments M_r, M_φ, and $M_{r\varphi}$ have the same values as the moments M_1, M_2, M_{12} (Fig. 54). Thus, transforming from Cartesian to polar co-ordinates by means of (11.20), and putting $\varphi = 0$ in formulas

(11.6), we have, finally,

$$M_r = -D\left(\frac{\partial^2 w}{\partial x_1^2} + \nu\frac{\partial^2 w}{\partial x_2^2}\right)_{\varphi=0} = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}\right)\right],$$

$$M_\varphi = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} + \nu\frac{\partial^2 w}{\partial r^2}\right), \qquad (11.23)$$

$$M_{\varphi r} = (1-\nu)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\varphi} - \frac{1}{r^2}\frac{\partial w}{\partial\varphi}\right).$$

In a similar way, from (11.10) we obtain formulas for the shearing forces

$$Q_r = -D\frac{\partial}{\partial r}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)w\right], \qquad (11.24)$$

$$Q_\varphi = -D\frac{1}{r}\frac{\partial}{\partial\varphi}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)w\right]. \qquad (11.25)$$

If the edge of a circular plate of radius a is clamped, then

$$w\Big|_{r=a} = 0, \qquad \frac{\partial w}{\partial r}\Big|_{r=a} = 0; \qquad (11.26)$$

if it is simply supported, then

$$w\Big|_{r=a} = M_r\Big|_{r=a} = 0; \qquad (11.27)$$

if it is free, then

$$M_r\Big|_{r=a} = 0, \qquad \left(Q_r - \frac{1}{r}\frac{\partial M_\varphi}{\partial\varphi}\right)_{r=a} = 0. \qquad (11.28)$$

The general solution of Eq. (11.22) is

$$w = w_0 + w_1,$$

where w_0 is a particular solution of Eq. (11.22), w_1 is the general solution of the homogeneous equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)w_1 = 0. \qquad (11.29)$$

The general solution of this equation is given by A. Clebsch in the form

$$w_1 = R_0^{(0)}(r) + \sum_{n=1}^{\infty} R_n^{(1)}(r)\cos n\varphi + \sum_{n=1}^{\infty} R_n^{(2)}(r)\sin n\varphi. \qquad (11.30)$$

The solution $R_0^{(0)}(r)$, which is independent of the angle φ, represents the symmetrical bending of a circular plate. Substituting this solution in Eq. (11.29) gives

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)\left(\frac{d^2 R_n^{(k)}}{dr^2} + \frac{1}{r}\frac{dR_n^{(k)}}{dr} - \frac{n^2}{r^2}R_n^{(k)}\right) = 0,$$

where $k = 0, 1, 2$. The general solution of this equation for $n = 0$ is

$$R_0^{(0)} = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r; \tag{11.31}$$

for $n = 1$

$$R_1^{(k)} = A_1^{(k)} r + B_1^{(k)} r^3 + C_1^{(k)} r^{-1} + D_1^{(k)} r \ln r; \tag{11.32}$$

for $n \geqslant 2$

$$R_n^{(k)} = A_n^{(k)} r^n + B_n^{(k)} r^{-n} + C_n^{(k)} r^{n+2} + D_n^{(k)} r^{n+2}. \tag{11.33}$$

The constants of integration $A_n^{(k)}$, $B_n^{(k)}$, $C_n^{(k)}$, and $D_n^{(k)}$ $(k = 1, 2)$ are determined from the fixing conditions for the edge of the plate.

87. SYMMETRICAL BENDING OF A CIRCULAR PLATE

Consider the transverse bending of a circular plate of radius a under a uniformly distributed load p when the plate is (1) simply supported along the edge and (2) clamped along the edge.

From the axial symmetry of the bending and from Clebsch's solution (11.30), the solution of the problem is sought in the form

$$w = w_0 + R_0^{(0)}(r),$$

where w_0 is a particular solution of the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w = \frac{p}{D}, \tag{11.34}$$

which follows from (11.22); this solution is given by

$$w_0 = \frac{p}{64D} r^4.$$

For the solution $R_0^{(0)}(r)$ determined by formula (11.31) to be bounded, we must take $C_0 = 0$; then

$$w = A_0 + B_0 r^2 + D_0 r^2 \ln r + \frac{p}{64D} r^4.$$

On the basis of formula (11.24) we have, at any section r,

$$Q_r = -D\left[\frac{p}{2D} r + 4D_0 \frac{1}{r}\right].$$

On the other hand, $Q_r = -\frac{p}{2} r$. Hence, $D_0 = 0$. Thus

$$w = A_0 + B_0 r^2 + \frac{p}{64D} r^4. \tag{11.35}$$

The coefficients A_0, B_0 are determined from the fixing condition for the plate along the edge. For the case (1) we have, when $r = a$,

$$w = 0, \quad \frac{\partial^2 w}{\partial r^2} + \nu \frac{1}{r}\frac{\partial w}{\partial r} = 0. \tag{11.36}$$

For the case (2), when $r = a$,

$$w = 0, \quad \frac{\partial w}{\partial r} = 0. \tag{11.37}$$

Substituting (11.35) in conditions (11.36) and (11.37), we obtain a system of linear algebraic equations:
for the case (1)

$$A_0 + B_0 a^2 + \frac{p}{64D} a^4 = 0,$$

$$2B_0 + \frac{3p}{16D} a^2 + \nu \left(2B_0 + \frac{p}{16D} a^2\right) = 0,$$

for the case (2)

$$A_0 + B_0 a^2 + \frac{p}{64D} a^4 = 0,$$

$$2B_0 a + \frac{p}{16D} a^3 = 0.$$

After determining the constants A_0, B_0, we finally obtain:
for the case (1)

$$w = \frac{p}{64D} (a^2 - r^2) \left(\frac{5+\nu}{1+\nu} a^2 - r^2\right),$$

for the case (2)

$$w = \frac{p}{64D} (a^2 - r^2)^2.$$

From the first formula of (11.4) for the case (1)

$$\sigma_{rr} = \frac{3(3+\nu)}{32} \frac{px_3}{h^3} (a^2 - r^2).$$

This stress at the centre of the plate ($r = 0$) is

$$\sigma_{rr} = \frac{3(3+\nu)}{32} \frac{a^2 px_3}{h^3}.$$

According to the exact solution, the stress at the centre of the plate is, by formula (9.77),

$$\sigma_{rr} = \frac{3(3+\nu)}{32} \frac{a^2 px_3}{h^3} + \frac{2+\nu}{8} p \left(\frac{x_3^3}{h^3} - \frac{3}{5} \frac{x_3}{h}\right).$$

Comparing the last two formulas, we notice that the additional term appearing in the exact solution is small if the thickness of the plate is small compared with the radius. Thus, when $\nu = 0.25$ and $x_3 = h$, for $\frac{2h}{a} = \frac{1}{20}$, $\frac{2h}{a} = \frac{1}{10}$, $\frac{2h}{a} = \frac{1}{5}$ the additional term is, respectively, 0.94, 3.8, and 15 per cent of the leading term.

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TO THE READER

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